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ANALYTIC GEOMETRY

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PREFACE

In adding another text-book on analytic geometry to the large list of such books now available, the authors feel that a statement of the reasons for this one is desirable. In compiling the work they have endeavored to keep constantly in view the needs of that large class of students in our colleges and technical schools who, though in a way sympathetic toward mathematical study, are yet lacking in that quick appreciation of the mathematical point of view which is a characteristic of the so-called mathematical mind. The assimilation of new ideas, or even of new combinations and applications of ideas already familiar, in which the study of analytic geometry especially abounds, is not easy even for the trained mind, and for the student approaching the subject for the first time is the source of most of his difficulties. The attempt here is to make this approach by easy gradations so that the student may not lose himself on the way, but may feel sure of his ground as he advances. To try to gain this end by the lack of rigorous demonstration would be to lose sight of an important object of such study, which, rightly viewed, is to assist the student to an acquaintance with correct principles of mathematical reasoning and the accompanying scientific attitude of mind. The aim has been therefore to preserve mathematical rigor throughout.

As to details, much of the book necessarily follows well established lines of procedure. The experienced teacher will find, however, here and there, some departure from these lines which it is hoped will be approved. Owing to the increasing use of the imaginary, and its growing importance to the student of both pure and applied mathematics, some elementary dis-

cussion of imaginary elements in geometry has been included, which the authors believe will be of value in accustoming the student to look upon the complex number as a useful member of the number system. The lists of exercises have been prepared with especial care. Throughout the text short lists have been inserted where needed for the immediate illustration of principles or methods. At the conclusion of each chapter longer lists are inserted. These are divided into two parts, " normal exercises," which are direct applications of the text of the chapter, and " general exercises," which are designed to give the student opportunity of testing his grasp of the subject as a whole, and of its underlying principles. There will be found in each set of " normal exercises " one or two examples illustrating each separate point of the theory developed in the corresponding portion of the text, so far as it can be illustrated by exercises. The student may therefore be assured that he has made application of all of the theory when he shall have worked *all* the normal exercises.

While the book is to be regarded as a text-book of plane analytic geometry, the concluding chapter is devoted to solid geometry. Only the barest outline of some of the fundamental principles of this subject is included, enough to enable the student, when he studies calculus, and wishes to apply its principles to problems involving solids and surfaces, to feel that he is not on entirely new ground.

University of Pennsylvania,
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FORMULAS, TABLES, ETC., FOR REFERENCE

A few important definitions, formulas, and tables, from elementary algebra and trigonometry, which will be needed in the study of this book, are placed here for convenience of reference.

A. Algebraic definitions and formulas

I. Definitions

(*a*) The **degree of a term** in specified letters is the number of times these letters occur as factors in that term. Thus ab^2c^3d is of the seventh degree in a, b, c, and d; $a^2x^2y^2$ is of the fourth degree in x and y.

(*b*) The **degree of a polynomial** in specified letters is defined as that of the term of highest degree in those letters. Thus $ax^3 + bx^2y + cx^2 + dy + e$ is of the third degree in x and y.

(*c*) A polynomial is said to be **homogeneous** in specified letters when each of its terms is of the same degree in those letters. Thus $ax^2 + 5xy - b^2y^2$ is homogeneous and of the second degree in x and y.

(*d*) In an equation containing one or more variables the term which contains no variable factors is called the **absolute term.** Thus in $ax^2 + by^2 - a^2c^2 = 0$, where x and y are the variables, the absolute term is $-a^2c^2$. In $x + y = 0$, the absolute term is zero.

(*e*) A **root of an equation** is a value of the variable which, when substituted for the variable, satisfies the equation.

II. The Quadratic Equation

(*a*) The roots of the quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

(*b*) The roots of the quadratic (1) are

$$\left.\begin{array}{ll} \text{real and unequal} & \text{if } b^2 > 4ac, \\ \text{real and equal} & \text{if } b^2 = 4ac, \\ \text{imaginary} & \text{if } b^2 < 4ac. \end{array}\right\} \tag{3}$$

(*c*) The sum of the roots of the quadratic (1) is

$$r_1 + r_2 = -\frac{b}{a}. \tag{4}$$

(*d*) The product of the roots of the quadratic (1) is

$$r_1 r_2 = \frac{c}{a}. \tag{5}$$

(*e*) If the coefficient a in the quadratic (1) varies and approaches zero, the coefficients b and c remaining constant, one root of the equation increases without limit.

III. Factors of $Ax^2 + Hxy + By^2$

A homogeneous expression of the second degree in x and y,

$$Ax^2 + Hxy + By^2, \tag{6}$$

can always be factored. The factors are conveniently expressed thus

$$\frac{1}{4A}[2Ax + (H + \sqrt{H^2 - 4AB})y][2Ax + (H - \sqrt{H^2 - 4AB})y]. \tag{7}$$

IV. Imaginaries

(*a*) $\sqrt{-1}$ is called the imaginary unit, and is usually represented by the letter i.* Thus $x + iy$ means $x + y\sqrt{-1}$.

(*b*) An expression containing both real and imaginary terms is called a complex expression.

(*c*) If two complex expressions are equal, the real parts of the two expressions are equal, and the imaginary parts are equal. Thus

$$\left.\begin{array}{lll} \text{if } x + iy = a + ib, & \text{then } x = a, & \text{and } y = b, \\ \text{if } x + iy = 0, & \text{then } x = 0, & \text{and } y = 0. \end{array}\right\} \tag{8}$$

V. Logarithms

(*a*) If $y = a^x$, x is called the logarithm of y to the base a. This equation may therefore be written in the form $x = \log_a y$.

(*b*) In the case of the so-called common logarithms, the logarithms used in computation, the base is 10.

* In electrical theory j is used instead of i to represent $\sqrt{-1}$.

(*c*) Logarithms used algebraically are usually Napierian, or natural logarithms. The base of this system of logarithms is the incommensurable number $e = 2.7182818285\cdots$, called the Napierian base, and defined by the infinite series

$$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 5} + \cdots. \tag{9}$$

(*d*) When in algebraic work a logarithm is written without any expressed base, e. g., $\log (1 + x^2)$, the base is understood to be e.

B. Trigonometric formulas

If l is the length, and r the radius of the arc subtending the angle of θ radians,

$$l = r\theta. \tag{10}$$

$$180^\circ = \pi \text{ radians}. \tag{11}$$

$$\csc \theta = \frac{1}{\sin \theta}, \qquad \cot \theta = \frac{1}{\tan \theta}, \qquad \sec \theta = \frac{1}{\cos \theta}. \tag{12}$$

$$\sin^2 \theta + \cos^2 \theta = 1. \tag{13}$$

$$\sec^2 \theta = 1 + \tan^2 \theta. \tag{14}$$

$$\csc^2 \theta = 1 + \cot^2 \theta. \tag{15}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}. \tag{16}$$

If $\tan \theta = \frac{m}{n}$, then

$$\sin \theta = \pm \frac{m}{\sqrt{m^2 + n^2}}, \qquad \cos \theta = \pm \frac{n}{\sqrt{m^2 + n^2}}. \tag{17}$$

$$\cos \theta = \sin (90^\circ \pm \theta), \qquad \cot \theta = \begin{cases} \tan (90^\circ - \theta), \\ -\tan (90^\circ + \theta). \end{cases} \tag{18}$$

$$\sin (180^\circ - \theta) = \sin \theta, \qquad \cos (180^\circ - \theta) = -\cos \theta,$$
$$\tan (180^\circ - \theta) = -\tan \theta. \tag{19}$$

$$\sin (180^\circ + \theta) = -\sin \theta, \qquad \cos (180^\circ + \theta) = -\cos \theta,$$
$$\tan (180^\circ + \theta) = \tan \theta. \tag{20}$$

$$\sin (-\theta) = -\sin \theta, \quad \cos (-\theta) = \cos \theta, \quad \tan (-\theta) = -\tan \theta. \tag{21}$$

$$\sin (x \pm y) = \sin x \cos y \pm \cos x \sin y. \tag{22}$$

$$\cos (x \pm y) = \cos x \cos y \mp \sin x \sin y. \tag{23}$$

$$\tan (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}. \tag{24}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta. \tag{25}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta. \tag{26}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}. \tag{27}$$

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta, \qquad \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta. \tag{28}$$

$$1 - \cos \theta = 2 \sin^2 \tfrac{1}{2}\theta, \qquad 1 + \cos \theta = 2 \cos^2 \tfrac{1}{2}\theta. \tag{29}$$

$$\tan \tfrac{1}{2}\theta = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}. \tag{30}$$

In any plane triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \tag{31}$$

$$a^2 = b^2 + c^2 - 2bc \cos A, \text{ etc.} \tag{32}$$

Signs of the Trigonometric Functions

	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
$\sin \theta$ $\csc \theta$	+	+	−	−
$\cos \theta$ $\sec \theta$	+	−	−	+
$\tan \theta$ $\cot \theta$	+	−	+	−

C. Values of the trigonometric functions of angles

I. *Certain exact values of these functions*

Angle in Degrees	Angle in Radians	sin	cos	tan	cot	sec	csc
0°	0	0	1	0	∞	1	∞
30°	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{4}\pi$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	$\frac{1}{2}\pi$	1	0	∞	0	∞	1

II. *Decimal values of the trigonometric functions for each 5° of the quadrant*

Angle in Degrees	Angle in Radians	sin	cos	tan	cot	sec	csc
0°	0	0.000	1.000	0.000	∞	1.000	∞
5°	0.087	0.087	0.996	0.087	11.430	1.004	11.474
10°	$0.17\bar{5}$	0.174	$0.98\bar{5}$	0.176	5.671	1.015	5.759
15°	0.262	0.259	0.966	0.268	3.732	1.035	3.864
20°	0.349	0.342	0.940	0.364	2.747	1.064	2.924
25°	0.436	0.423	0.906	0.466	$2.14\bar{5}$	1.103	2.366
30°	0.524	0.500	0.866	0.577	1.732	$1.15\bar{5}$	2.000
35°	0.611	0.574	0.819	0.700	1.428	1.221	1.743
40°	0.698	0.643	0.766	0.839	1.192	1.305	1.556
45°	0.785	0.707	0.707	1.000	1.000	1.414	1.414
50°	0.873	0.766	0.643	1.192	0.839	1.556	1.305
55°	0.960	0.819	0.574	1.428	0.700	1.743	1.221
60°	1.047	0.866	0.500	1.732	0.577	2.000	$1.15\bar{5}$
65°	1.134	0.906	0.423	$2.14\bar{5}$	0.466	2.366	1.103
70°	1.222	0.940	0.342	2.747	0.364	2.924	1.064
75°	1.309	0.966	0.259	3.732	0.268	3.864	1.035
80°	1.396	$0.98\bar{5}$	0.174	5.671	0.176	5.759	1.015
85°	1.484	0.996	0.087	11.430	0.087	11.474	1.004
90°	1.571	1.000	0.000	∞	0.000	∞	1.000

D. Miscellaneous tables

I. *Degrees and Radians*

$1° = 0.0174533$ rad., $1' = 0.0002909$ rad., $1'' = 0.0000048$ rad.

1 rad. $= 57°.29578 = 57°\ 17'\ 44''.8 = 3437'.75 = 206264''.8$

0.1 rad. = 5° 43′ 46″.5	0.4 rad. = 22° 55′ 5″.9	0.7 rad. = 40° 6′ 25″.4
0.2 " = 11° 27′ 33″.0	0.5 " = 28° 38′ 52″.4	0.8 " = 45° 50′ 11″.8
0.3 " = 17° 11′ 19″.4	0.6 " = 34° 22′ 38″.9	0.9 " = 51° 33′ 58″.3

II. *Mantissas of the common logarithms of numbers*

N	0	1	2	3	4	5	6	7	8	9
1	000	041	079	114	146	176	204	230	255	279
2	301	322	342	362	380	398	41$\bar{5}$	431	447	462
3	477	491	505	519	531	544	556	568	580	591
4	602	613	623	633	643	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	833	839
7	845	851	857	863	869	875	881	886	892	898
8	903	908	914	919	924	929	934	940	944	949
9	954	959	964	968	973	978	982	987	991	996

III. *Napierian logarithms of numbers*

N	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	$-\infty$	−2.30	−1.61	−1.20	−0.92	−0.69	−0.51	−0.36	−0.22	−0.11
1	0.00	0.10	0.18	0.26	0.34	0.41	0.47	0.53	0.59	0.64
2	0.69	0.74	0.79	0.83	0.88	0.92	0.96	0.99	1.03	1.06
3	1.10	1.13	1.16	1.19	1.22	1.25	1.28	1.31	1.34	1.36
4	1.39	1.41	1.44	1.46	1.48	1.50	1.53	1.5$\bar{5}$	1.57	1.59
5	1.61	1.63	1.6$\bar{5}$	1.67	1.69	1.70	1.72	1.74	1.76	1.77
6	1.79	1.81	1.82	1.84	1.86	1.87	1.89	1.90	1.92	1.93
7	1.9$\bar{5}$	1.96	1.97	1.99	2.00	2.01	2.03	2.04	2.05	2.07
8	2.08	2.09	2.10	2.12	2.13	2.14	2.15	2.16	2.17	2.19
9	2.20	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29
10	2.30	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39

IV. *Positive powers of e (Napierian anti-logarithms)*

Exp.	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.00	1.11	1.22	1.3$\bar{5}$	1.49	1.6$\bar{5}$	1.82	2.01	2.23	2.46
1	2.72	3.00	3.32	3.67	4.06	4.48	4.95	5.47	6.0$\bar{5}$	6.69
2	7.39	8.17	9.03	9.97	11.0	12.2	13.$\bar{5}$	14.9	16.4	18.2
3	20.1	22.2	24.5	27.1	30.0	33.1	36.6	40.4	44.7	49.4
4	54.6	60.3	66.7	73.7	81.$\bar{5}$	90.0	99.$\bar{5}$	110.	122.	134.
5	148.	164.	181.	200.	221.	24$\bar{5}$.	270.	299.	330.	365.
6	403.	446.	493.	54$\bar{5}$.	602.	665.	735.	812.	898.	992.

V. *Negative powers of e (Napierian anti-logarithms)*

Exp.	0	−.1	−.2	−.3	−.4	−.5	−.6	−.7	−.8	−.9
−5	0.01	0.01	0.01	0.00						
−4	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
−3	0.0$\bar{5}$	0.0$\bar{5}$	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.02
−2	0.14	0.12	0.11	0.10	0.09	0.08	0.07	0.07	0.06	0.06
−1	0.37	0.33	0.30	0.27	0.2$\bar{5}$	0.22	0.20	0.18	0.17	0.1$\bar{5}$
0	1.00	0.90	0.82	0.74	0.67	0.61	0.5$\bar{5}$	0.50	0.4$\bar{5}$	0.41

VI. *Square roots of numbers*

N	0	1	2	3	4	5	6	7	8	9
0	000	1.00	1.41	1.73	2.00	2.24	2.4$\bar{5}$	2.6$\bar{5}$	2.83	3.00
1	3.16	3.32	3.46	3.61	3.74	3.87	4.00	4.12	4.24	4.36
2	4.47	4.58	4.69	4.80	4.90	5.00	5.10	5.20	5.29	5.39
3	5.48	5.57	5.66	5.74	5.83	5.92	6.00	6.08	6.16	6.24
4	6.32	6.40	6.48	6.56	6.63	6.71	6.78	6.86	6.93	7.00
5	7.07	7.14	7.21	7.28	7.3$\bar{5}$	7.42	7.48	7.5$\bar{5}$	7.62	7.68
6	7.7$\bar{5}$	7.81	7.87	7.94	8.00	8.06	8.12	8.19	8.2$\bar{5}$	8.31
7	8.37	8.43	8.49	8.54	8.60	8.66	8.72	8.77	8.83	8.89
8	8.94	9.00	9.06	9.11	9.17	9.22	9.27	9.33	9.38	9.43
9	9.49	9.54	9.59	9.64	9.70	9.7$\bar{5}$	9.80	9.8$\bar{5}$	9.90	9.9$\bar{5}$

The Greek alphabet

Α, α, alpha	Ι, ι, iota	Ρ, ρ, rho
Β, β, beta	Κ, κ, kappa	Σ, σ, ς, sigma
Γ, γ, gamma	Λ, λ, lambda	Τ, τ, tau
Δ, δ, delta	Μ, μ, mu	Υ, υ, upsilon
Ε, ϵ, epsilon	Ν, ν, nu	Φ, ϕ, phi
Ζ, ζ, zeta	Ξ, ξ, xi	Χ, χ, chi
Η, η, eta	Ο, o, omicron	Ψ, ψ, psi
Θ, θ, ϑ, theta	Π, π, pi	Ω, ω, omega.

ANALYTIC GEOMETRY

CHAPTER I

CARTESIAN COORDINATES

1. Theorem. *Let $X'OX$, $Y'OY$ be two fixed straight lines in a plane, then the position of any point in the plane is determined by its distance from $Y'Y$ measured parallel to $X'X$, and its distance from $X'X$ measured parallel to $Y'Y$.*

Let P be a point in the plane of $X'X$, $Y'Y$, whose distance to the right of $Y'Y$, measured parallel to $X'X$ is a, and whose distance above $X'X$, measured parallel to $Y'Y$ is b. Then the position of P in the plane is determined. Because P must lie in the line MN, parallel to $Y'Y$ at the distance a (as defined above) from it on the right, and also in the line RS, parallel to $X'X$ at the distance b from it on the upper side. Hence P must be at the intersection of MN and RS.

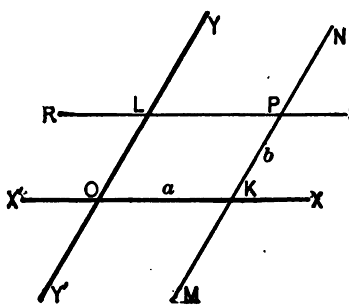


Fig. 1

Definitions. The two lengths or distances which determine the position of a point, as a and b determine P in the preceding demonstration, are called the **coordinates of the point.**

The fixed lines in the plane to which the position of the point is referred are called the **axes of coordinates,** and their intersection O is called the origin of coordinates, or simply the

origin. The axis $X'X$ is called the **axis of abscissas,** or the **X-axis,** and $Y'Y$ is called the **axis of ordinates,** or the **Y-axis.**

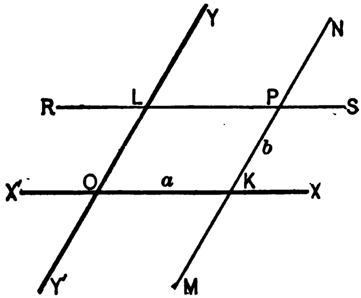


FIG. 1

The two coordinates a and b of the point P, Fig. 1, are called respectively the **abscissa** and the **ordinate** of the point. That is:

The abscissa of a point is its distance from the Y-axis, measured parallel to the X-axis.

The ordinate of a point is its distance from the X-axis, measured parallel to the Y-axis.

The letter x is customarily used to designate the abscissa of a point, and the letter y to designate the ordinate. So also the expression "the x of a point" is often used, meaning its abscissa, or "the y of a point," meaning its ordinate.

The coordinate axes are usually taken at right angles to each other, and in this book they will always be considered as so situated unless the contrary is stated. When the axes are not at right angles the coordinate system is said to be oblique.

NOTE. The term *Cartesian coordinates* is applied to the system of coordinates which has just been described because it is essentially the system invented by the French philosopher and mathematician Descartes (1596–1650). Other coordinate systems are often used, one of which, the polar system, will be described in a later chapter of this book.

2. Signs of the coordinates. Notation. In order to be able to locate a point when its coordinates are given we must know not only the lengths of its coordinates but also the direction (right or left, up or down) in which they are to be measured. Thus, Fig. 2, if $NO = OM = a$, and $RO = OS = b$, there are four points P_1, P_2, P_3, P_4 at the distance a from $Y'Y$ and the

distance b from $X'X$. To avoid this ambiguity we adopt the usual convention of distinguishing between lengths measured in opposite directions by $+$ and $-$ signs. Then, Fig. 3,

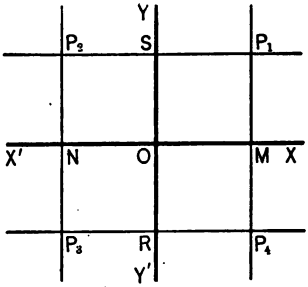


FIG. 2

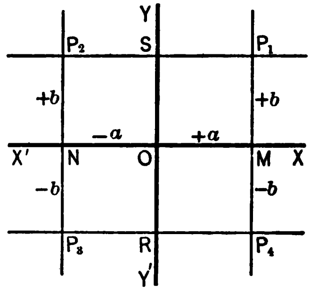


FIG. 3

if OM is called $+\ a$, ON is $-\ a$; and if OS is $+\ b$, OR is $-\ b$. Hence the coordinates of P_1 are $x = +\ a$, $y = +\ b$, those of P_2 are $x = -\ a$, $y = +\ b$, those of P_3 are $x = -\ a$, $y = -\ b$, and those of P_4 are $x = +\ a$, $y = -\ b$.

In actual practice when an abscissa or ordinate is $+$ the sign is usually not written.

In designating the position of a point by its coordinates the notation (x, y) is used. That is, the coordinates are written in parenthesis with a comma separating them, and the abscissa is invariably written first. Thus

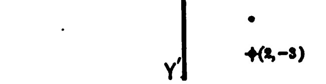


FIG. 4

the point $(2, -3)$ means the point whose abscissa is 2 and ordinate -3, the point $(-3, 2)$ is the point whose abscissa is -3 and ordinate 2. See Fig. 4.

EXERCISES. 1. Draw a pair of coordinate axes and, using any

convenient unit of length, locate the following points (1, 2), (5, −2), (3, 3), (− 3, 1), (− 2, 0), (0, 4), (− 4, − 2), (3, 0), (0, − 2), (0, 0).

2. If a point is on the X-axis what is its ordinate?
3. If a point is on the Y-axis what is its abscissa?
4. What are the coordinates of the origin?
5. If the abscissa of a point is 0, on what line will it lie?
6. If the ordinate of a point is 0, on what line will it lie?

3. Directed lengths. By a directed length is meant a length measured in a given direction. As has been explained the coordinates of a point are always directed lengths. The length AB means the distance from A to B in magnitude and direction.

I. If A and B are two points, then

length AB = − length BA.

FIG. 5

That is, the magnitude element of the length is the same, whether we think of AB or of BA, but the direction element in the one case is opposite to that in the other.

II. THEOREM. *If three points A, B, C are arranged in any order on a straight line then $AC = AB + BC$.*

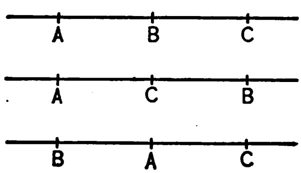


FIG. 6

In the upper line, Fig. 6, the three segments AC, AB, BC are all measured in the same direction, and AC is composed of the other two. Hence $AC = AB + BC$.

In the second line of the figure $AC = AB - CB$, but from I, $-CB = +BC$, hence $AC = AB + BC$.

In the third line $AC = BC - BA = AB + BC$, as before.

Other possible arrangements can be similarly treated.

III. Problem. *To express the distance from one point to another, when the distances of these two points from a third point on the same line are known.*

Let the distances AB and AC, Fig. 6, be known in both length and direction, and let the distance from B to C be required. It is then only necessary to express the length of the required segment in terms of the given segments as in II, being careful to write the letters in the proper order in each case. Since it is here required to find the distance *from B to C* we write BC, and then, according to II,

$$BC = BA + AC,$$

and hence

$$BC = -AB + AC. \tag{1}$$

Had the question been to find the distance from C to B, the same procedure would lead to the equation

$$CB = CA + AB,$$

or

$$CB = -AC + AB. \tag{2}$$

The student should examine this discussion in connection with each arrangement of points in Fig. 6 and satisfy himself that the truth of the result is independent of the order of the points.

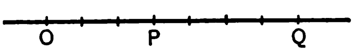


Fig. 7

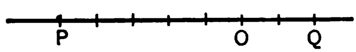


Fig. 8

Illustrations. In Fig. 7 let $OP = 3$, and $OQ = 7$, then

$$PQ = PO + OQ = -3 + 7 = 4.$$

Again in Fig. 8 suppose $OP = -5$, and $OQ = 2$, then

$$PQ = PO + OQ = -(-5) + 2 = 7.$$

Similarly with the same data

$$QP = QO + OP = -2 + (-5) = -7.$$

4. Distance between two points.

PROBLEM. *To find the length of a line joining two points whose coordinates are given, the axes being rectangular.*

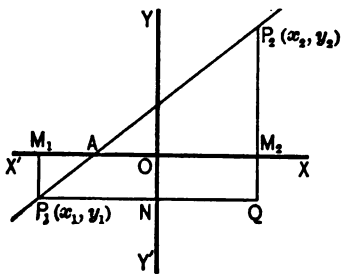


FIG. 9

Let it be required to express the length of P_1P_2, in terms of the given coordinates (x_1, y_1), (x_2, y_2) of these two points. That is we have given $OM_1 = x_1$, $M_1P_1 = y_1$, $OM_2 = x_2$, $M_2P_2 = y_2$.

Draw P_1Q parallel to OX to meet M_2P_2 (extended in this case) at Q. Since P_1QP_2 is a right angle,

$$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{QP_2}^2. \tag{i}$$

But

$$P_1Q = P_1N + NQ = -OM_1 + OM_2 = -x_1 + x_2,$$

and

$$QP_2 = QM_2 + M_2P_2 = -M_1P_1 + M_2P_2 = -y_1 + y_2.$$

Substituting these values in (*i*) the result is

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

or since $(x_2 - x_1)^2 = (x_1 - x_2)^2$, and $(y_2 - y_1)^2 = (y_1 - y_2)^2$,

$$\boldsymbol{P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}. \tag{3}$$

Note carefully that in indicating a line-segment, simply as a length, it is to be written in the positive direction, as P_1Q, QP_2 in the foregoing demonstration; but in indicating a line-segment as one of the coordinates of a point it must be taken as starting *from* the axis and going *to* the point. Thus, above, OM_1 (not M_1O) is x_1, and M_1P_1 (not P_1M_1) is y_1.

5. Slope of a line.

I. DEFINITION. *The tangent of the angle which a line makes with the X-axis is called the* **slope** *of the line.*

Of the four angles formed by the intersection of a line with the X-axis that one is taken as the " slope " angle which lies between the positive extension of the X-axis and the upper extension of the line. Thus the slope of AB is $\tan \theta_1$ and the slope of CD is $\tan \theta_2$.

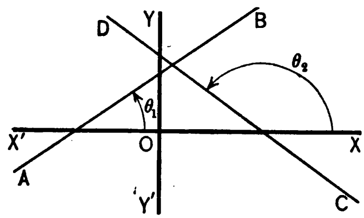


FIG. 10

II. PROBLEM. *To find the slope of a line in terms of the coordinates of two points on the line.*

The slope of the line determined by the points P_1, P_2 Fig. 9 is $\tan XAP_2 = \tan QP_1P_2$. Now

$$\tan QP_1P_2 = \frac{QP_2}{P_1Q},$$

and as shown in Art. 4, $QP_2 = -y_1 + y_2$, and $P_1Q = -x_1 + x_2$,

$$\therefore \quad \tan QP_1P_2 = m = \frac{-y_1 + y_2}{-x_1 + x_2} = \frac{y_1 - y_2}{x_1 - x_2}, \qquad (4)$$

where m is used to designate the slope.

Since in deriving (3) and (4) the principles of directed lengths have been strictly followed the results are true for all positions of the points P_1 and P_2.

6. Division of a segment of a line in a given ratio.

PROBLEM. *To find the coordinates of the point which divides a given segment of a line in a given ratio.*

Some preliminary remarks are necessary.

(*a*) By a given segment of a line is meant a segment the coordinates of whose extremities are known.

(*b*) A segment may be divided internally or externally. Thus in Fig. 11 (*a*) the point P divides the segment AB in-

ternally in the ratio $AP : PB$; and in Fig. 11 (*b*) the point P

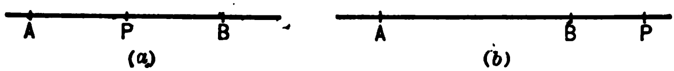


FIG. 11

divides the segment AB externally in the ratio $AP : PB$.

(*c*) When the division is *internal* the ratio is *positive*, because the two segments AP, PB, which are the terms of the ratio, are measured in the same direction, and hence have the same sign; but when the division is *external* the ratio is negative because in that case the two segments AP, PB are measured in opposite directions.

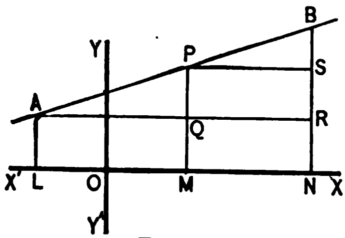


FIG. 12

Our problem then is to find the coordinates of P, having given the coordinates of A and B, and the ratio $AP : PB$.

Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $AP : PB = h : k$. Draw LA, MP, NB parallel to OY, and PS, AR parallel to OX. Then

$$\frac{AQ}{PS} = \frac{AP}{PB} = \frac{h}{k}, \tag{i}$$

$$\frac{QP}{SB} = \frac{AP}{PB} = \frac{h}{k}. \tag{ii}$$

But

$$\frac{AQ}{PS} = \frac{LM}{MN} = \frac{LO + OM}{MO + ON} = \frac{-x_1 + x}{-x + x_2}, \tag{iii}$$

$$\frac{QP}{SB} = \frac{QM + MP}{SN + NB} = \frac{AL + MP}{PM + NB} = \frac{-y_1 + y}{-y + y_2}. \tag{iv}$$

Hence, substituting from (*iii*) in (*i*) and from (*iv*) in (*ii*)

$$\frac{-x_1 + x}{-x + x_2} = \frac{h}{k}, \quad \frac{-y_1 + y}{-y + y_2} = \frac{h}{k}. \tag{v}$$

Solving the first of (v) for x,

$$-kx_1 + kx = -hx + hx_2,$$

or

$$(h + k)x = hx_2 + kx_1,$$

and

$$\left.\begin{aligned} x &= \frac{hx_2 + kx_1}{h + k}. \\ \text{Similarly} \qquad y &= \frac{hy_2 + ky_1}{h + k}. \end{aligned}\right\} \tag{5}$$

It should be noted that equations (5) are true whether the axes are rectangular or oblique. Also that when P divides AB externally, the ratio being then negative, it is immaterial, in applying (5), whether the negative sign is attached to h or k.

IMPORTANT SPECIAL CASE. The case of bisection is especially important. In this case $h : k = 1 : 1$, and substituting $h = k = 1$ in (5) we have

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}, \tag{6}$$

which are the coordinates of the point bisecting the line joining (x_1, y_1) and (x_2, y_2).

EXERCISES. 1. Find the length of the sides of the triangle whose vertices are the points $(2, 3)$, $(-2, 0)$, $(1, -4)$. Ans. $5, 5, 5\sqrt{2}$.

2. Find the length of the sides and diagonals of the quadrilateral whose vertices are the points $(5, 2)$, $(2, 6)$, $(-3, 4)$, $(-1, -1)$.
Ans. Sides 5, $\sqrt{29}$, $\sqrt{29}$, $3\sqrt{5}$; diagonals $\sqrt{68}$, $\sqrt{58}$.

3. Find the middle points of the sides of the triangle whose vertices are $(5, 2)$, $(-1, 6)$, $(1, -4)$, and find the lengths of the medians of the triangle. Ans. $\sqrt{26}$, $\sqrt{65}$, $\sqrt{65}$.

4. What form does formula (3) take if one of the two given points is the origin?

5. Prove that the quadrilateral whose vertices are $(4, 2)$, $(-2, -1)$, $(0, -4)$, $(6, -1)$ is a parallelogram.

6. Find the coordinates of both points of trisection of the line joining the two points $(1, 6)$, $(4, -2)$. Ans. $(2, \frac{10}{3})$, $(3, \frac{2}{3})$.

7. $A = (-2, 1)$, $B = (4, 3)$, **and** P **divides** AB **externally so that** $AP : PB = -2 : 5$. **Find the coordinates of** P. **Ans.** $(-6, -\frac{1}{3})$.

8. Trisect each median of the triangle in Ex. 3, so that the longer segment in each case is adjacent to the corresponding vertex. What does the result show?

9. Extend the line joining $P_1 = (-1, 3)$, $P_2 = (3, 5)$ **each way a distance equal to** P_1P_2, **and find the coordinates of the two points thus determined.** **Ans.** $(7, 7)$, $(-5, 1)$.

10. A parallelogram has two opposite vertices at (1, 2) and (4, 1), and a third vertex at the origin. Find the coordinates of the fourth vertex. **Ans.** $(5, 3)$.

7. Area of a triangle. Let the vertices of a triangle be

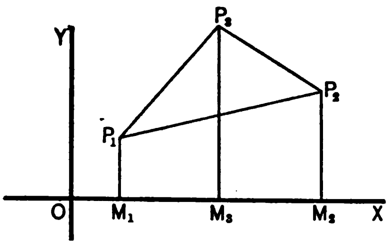


FIG. 13

$$P_1 = (x_1, y_1),$$
$$P_2 = (x_2, y_2),$$
$$P_3 = (x_3, y_3).$$

Then area $P_1P_2P_3 =$

$$M_1M_3P_3P_1 + M_3M_2P_2P_3 - M_1M_2P_2P_1. \quad (i)$$

Since each of the areas on the right in (i) is a trapezoid we have

$$\text{area } M_1M_3P_3P_1 = \tfrac{1}{2}M_1M_3(M_1P_1 + M_3P_3) = \tfrac{1}{2}(x_3 - x_1)(y_3 + y_1),$$
$$\text{area } M_3M_2P_2P_3 = \tfrac{1}{2}M_3M_2(M_3P_3 + M_2P_2) = \tfrac{1}{2}(x_2 - x_3)(y_2 + y_3),$$
$$\text{area } M_1M_2P_2P_1 = \tfrac{1}{2}M_1M_2(M_1P_1 + M_2P_2) = \tfrac{1}{2}(x_2 - x_1)(y_1 + y_2).$$

Hence substituting in (i)

$$\text{area } P_1P_2P_3 = \tfrac{1}{2}[(x_3 - x_1)(y_3 + y_1) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)],$$

from which, expanding and collecting terms,

$$\textbf{Area } \boldsymbol{P_1P_2P_3 = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]}. \quad (7)$$

Formula (7) may also be written thus:

$$\text{area } P_1P_2P_3 = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]. \quad (8)$$

When the coordinates of the vertices of a given triangle are substituted in (7) or (8) the resulting number expressing the area of this triangle will be positive or negative according to the order in which the points are taken. The numerical value obtained, irrespective of sign, will always be the required area.

EXAMPLE. Find the area of the triangle whose vertices are $(-1, 2)$, $(3, -1)$, $(-4, 2)$.

Substituting the given coordinates in (7) the result is

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}[-(-1-2) + 3(2-2) - 4(2+1)] \\ &= \tfrac{1}{2}(3 + 0 - 12) = -4\tfrac{1}{2}. \end{aligned}$$

The required area is therefore $4\frac{1}{2}$.

The student acquainted with the determinant notation will recognize that formula (8) can be written

$$\text{area } P_1P_2P_3 = \tfrac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}. \quad (9)$$

8. Area of any polygon. Since any polygon can be divided into triangles by diagonals drawn from one of the vertices, the area of such a figure can be determined by finding the areas of its several parts by means of (7), (8) or (9), and combining the results.

EXERCISES. 1. Determine the areas of the following triangles

(*a*) $(2, 6)$, $(4, -1)$, $(0, 0)$.
(*b*) $(3, -2)$, $(-4, 1)$, $(0, 5)$.
(*c*) $(1, -3)$, $(4, 5)$, $(-2, 3)$.
(*d*) $(0, 3)$, $(4, -5)$, $(-3, -2)$.
(*e*) $(5, -1)$, $(-1, -5)$, $(2, 4)$.
(*f*) $(3, -2)$, $(0, 4)$, $(5, 1)$.

Ans. (*a*) 13, (*b*) 20, etc.

2. Determine the area of the quadrilateral whose vertices are $(1, 4)$, $(5, -2)$, $(0, -3)$, $(-2, 0)$. Ans. $25\frac{1}{2}$.

3. Determine the area of the quadrilateral whose vertices are $(5, 2)$, $(2, -5)$, $(-5, 3)$, $(0, 1)$. Ans. 29.

EXERCISES* ON CHAPTER I

Normal Exercises

1. If the abscissa of a point is 4 on what line will it lie?

2. If the abscissa of a point is equal to its ordinate on what line will it lie?

3. What is the distance of the point (2, 4) from the origin?

4. How far is the point (− 3, − 5) from the point (− 1, 2)?

5. What is the slope of the line joining the point (2, 4) to the origin?

6. What is the slope of the line joining the points (2, − 3) and (− 4, − 2)?

7. Find the coordinates of the middle points of the sides of the triangle whose vertices are the points (0, 0), (− 2, − 4), (− 2, 3).

8. If the line from (− 2, 3) to (4, − 1) be extended through the latter point until its length is three times its original length, what are the coordinates of the end point? Ans. (16, − 9).

9. Find the area of the triangle of exercise 7.

10. Find the area of the quadrilateral whose vertices are (− 1, 3), (− 2, 3), (− 3, − 3), and (3, − 4). Ans. 22.

General Exercises

11. A straight line joining two points is bisected by the origin. One of the points is (2, 3). What is the other?

12. Determine which of the following sets of points are points of the same straight line

(*a*) (2, 4), (− 1, 0), (5, 8); (*c*) (2, 4), (3, 5), (5, 7), (0, 2);
(*b*) (− 1, 2), (0, − 1), (2, 4); (*d*) (− 1, 1), (2, − 1), (0, 0), (− 4, 3).

13. Show that (− 1, 2), (2, − 2), (4, 7), and (9, 5) are the vertices of a trapezoid.

* The exercises at the end of each chapter are divided into two groups, "Normal Exercises," and "General Exercises." Those given under the first heading require for their solution only the direct application of the methods or formulas developed in the chapter. Under the second heading are given other exercises of like character together with more varied problems requiring indirect, combined, or extended applications of the chapter's results.

14. Show that $(-1, 2)$, $(3, -1)$, $(4, 7)$, and $(8, 4)$ are the vertices of a parallelogram.

15. Show that $(2, 4)$, $(7, 3)$, $(-2, -3)$, $(-1, 2)$ are the vertices of an isosceles trapezoid.

16. Determine which of the following sets of points are the vertices of a parallelogram

(*a*) $(2, 5)$, $(0, 0)$, $(-1, -2)$, $(2, -4)$; (*b*) $(1, 2)$, $(7, 5)$, $(3, 3)$, $(5, 4)$.

17. Show that the middle points of the sides of the quadrilateral whose vertices are the points $(1, -2)$, $(2, 4)$, $(-1, -4)$, and $(3, -5)$ are the vertices of a parallelogram.

18. Find the coordinates of the fourth vertex of each of the parallelograms three of whose vertices are the points $(-1, 2)$, $(-3, 4)$, $(2, 1)$. Ans. $(0, 3)$, $(-6, 5)$, $(4, -1)$.

19. Find the lengths of the sides of the triangle whose vertices are the points $(2, 1)$, $(5, 5)$, and $(-5, 0)$. Ans. 5, $5\sqrt{2}$, $5\sqrt{5}$.

20. Show that the points $(-1, 3)$, $(-2, -4)$, $(6, -4)$, $(2, -6)$, and $(7, -1)$ are all on one circle whose center is the point $(2, -1)$.

21. Show that the diagonals of the quadrilateral whose vertices are the points $(0, 3)$, $(3, 5)$, $(5, 2)$, and $(2, 0)$ are equal. Show that this quadrilateral is a rectangle.

22. Find the point on the X-axis equally distant from the points $(2, 4)$ and $(-2, 6)$. Ans. $(-\frac{5}{2}, 0)$.

23. Find the point on the Y-axis equally distant from the points $(2, 4)$ and $(6, 5)$. Ans. $(0, 20\frac{1}{2})$.

24. Find the point equally distant from the points $(-2, -4)$, $(6, -4)$, and $(-1, 3)$. Ans. $(2, -1)$.

25. Show that the points $(-1, 2)$, $(4, -3)$, $(5, 3)$ are the vertices of an isosceles triangle.

26. Find the third vertex of one of the equilateral triangles two of whose vertices are the points $(1, 2)$, $(4, 6)$. Ans. $(\frac{5}{2} + 2\sqrt{3},\ 4 - \frac{3}{2}\sqrt{3})$.

27. The coordinates of the points A and B are $(-1, 2)$ and $(4, -2)$ respectively. P is a point on AB produced, and is twice as far from A as it is from B. Find its coordinates. Ans. $(9, -6)$.

28. The point $(4, 7)$ is on the line joining $(-1, 2)$ to the point B, and is one third as far from $(-1, 2)$ as from B. Find the coordinates of B. Ans. $(19, 22)$ or $(-11, -8)$.

29. Find the area of the triangles the coordinates of whose vertices are:

(*a*) (0, 0), (0, 4), (4, 0); (*b*) $(-1, -4)$, $(-5, -5)$, $(-2, -1)$.

30. Determine two values of x so that the area of the triangle whose vertices are the points $(x, 2)$, $(2x, 1)$, and (4, 5) is one half the area of the triangle whose vertices are the points $(x, -1)$, (2, 3), and (5, 2).

Ans. $-\frac{22}{3}$, $\frac{6}{5}$.

31. Show that the area of the triangle whose vertices are $(-3, 5)$, (3, 7), $(5, -1)$ is four times the area of the triangle whose vertices are the mid-points of its sides.

32. Find the area of the quadrilateral whose vertices are (4, 1), $(-1, 2)$, $(-2, -3)$, and (1, 3). Ans. $16\frac{1}{2}$.

33. Find the area of the polygon whose vertices are (2, 4), (5, 1), $(4, -3)$, $(1, -5)$, $(-2, -3)$, $(-2, 4)$; first by combining the areas of the triangles into which it is divided by diagonals from one vertex, and secondly, by combining the areas of the triangles formed by joining the vertices to the origin.

34. Find the coordinates of the middle points of each of the diagonals of the parallelogram whose vertices are (0, 0), $(a, 0)$, (b, c), and $(a + b, c)$, and thus prove that the diagonals of a parallelogram bisect each other.

Note. Exercises 35 to 40 inclusive are propositions in elementary plane geometry which are to be proved by the use of coordinates in a manner similar in general to that indicated in exercise 34.

35. Prove that the diagonals of a rectangle are equal to each other.

36. Prove that the mid-point of the hypotenuse of a right triangle is equally distant from each of the vertices.

37. The line joining the mid-points of two sides of a triangle is parallel to the third side and is half the length of the third side.

38. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

39. The lines joining the mid-points of opposite edges of any quadrilateral bisect each other.

40. The medians of a triangle intersect in one point which trisects each of them.

CHAPTER II

GRAPHICAL REPRESENTATION OF EQUATIONS

9. Variables and constants.

I. Definition. *A* **variable** *is a number whose value may change, arbitrarily, or in accordance with some law.*

The speed of a train changes as it gathers headway on leaving a station, hence the number which expresses the speed is a variable. A stone thrown into the air changes its distance from the ground from moment to moment, hence the number which expresses this distance is a variable.

II. Definition. *A number which does not change in value in the course of any discussion is called a* **constant.**

In analytic geometry the equations employed contain variables and constants. The constants are either definite numbers, as 3, $\frac{5}{8}$, π, log 2, $\sqrt{2}$, etc., or they may be represented by letters which stand for quantities whose values are assumed to be known in the problem under discussion.

10. Graphical representation of equations. Examples.

1. Let $x + y = 2$ be an equation in two variables x and y. The equation states in algebraic language that two variable numbers, x and y, are so related to each other that their sum is always 2. Evidently any value may be assigned to either x or y, and then the value of the other is determined. Thus if $x = 3$, $y = -1$; if $x = 1$, $y = 1$; if $x = -4$, $y = 6$; if $x = \frac{5}{2}$, $y = -\frac{1}{2}$; etc. If this be done systematically and the results arranged in a table we have, taking at first only integral values:

$x = -4$	-3	-2	-1	0	1	2	3	4	etc.
$y = 6$	5	4	3	2	1	0	-1	-2	

Draw a pair of coordinate axes and locate the points which have these corresponding pairs of values of x and y as abscissas and ordinates respectively, thus obtaining the points P_1, P_2, $\cdots$, P_9, Fig. 14 (*a*). Through the points thus obtained draw a line, as in Fig. 14 (*b*). This is found to be a straight line. As

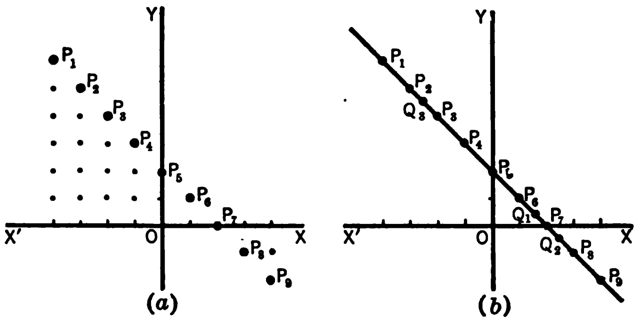


FIG. 14

many additional pairs of values of x and y as we please can be found from the equation, and the corresponding points located. Thus $x = \frac{3}{2}$, $y = \frac{1}{2}$ satisfy the equation and give the point Q_1; $x = 1 + \sqrt{2}$, $y = 1 - \sqrt{2}$ satisfy the equation and give the point Q_2; $x = -\sqrt{7}$, $y = 2 + \sqrt{7}$ satisfy the equation and give the point Q_3.

If it were possible to construct all of the infinitely many points whose coordinates satisfy the equation, we should find that they all lie on the line which has been drawn. This straight line is therefore a graphical representation (called briefly the **graph**) of the equation $x + y = 2$.

2. Construct the graph of $x^2 - 6x - 4y - 5 = 0$.

First solve the equation for y, thus

$$y = \tfrac{1}{4}(x^2 - 6x - 5).$$

Assigning values to x, and computing the corresponding values of y, the results are as follows:

$x =$	-2	-1	0	1	2	3	4	5	6	7	8
$y =$	$\frac{11}{4}$	$\frac{1}{2}$	$-\frac{5}{4}$	$-\frac{5}{2}$	$-\frac{13}{4}$	$-\frac{7}{2}$	$-\frac{13}{4}$	$-\frac{5}{2}$	$-\frac{5}{4}$	$\frac{1}{2}$	$\frac{11}{4}$
point	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}

Plotting the points thus determined it is evident that they do not lie on a straight line as was the case in example 1. If through the points thus located a curve be sketched smoothly the result is as shown in Fig. 15.

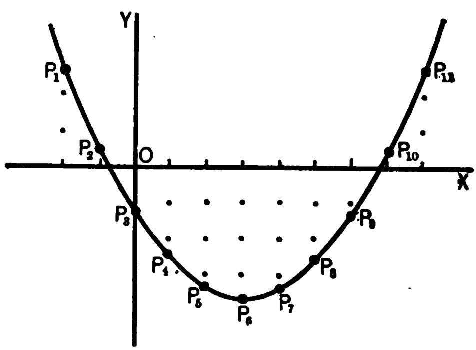


FIG. 15

3. Construct the graph of $y = x^3 - 6x^2 + 11x - 3$.

$x =$	-1	0	1	2	3	4	5
$y =$	-21	-3	3	3	3	9	27
point	*	P_1	P_2	P_3	P_4	P_5	*

* The points $(-1, -21)$ and $(5, 27)$ are omitted because the numerical values of y are too large to go on the diagram conveniently.

The five points P_1, P_2, $\cdots$, P_5 are shown by heavy dots in Fig. 16 (*a*). From these alone the true shape of the graph is uncertain, and some intermediate points must be found. When

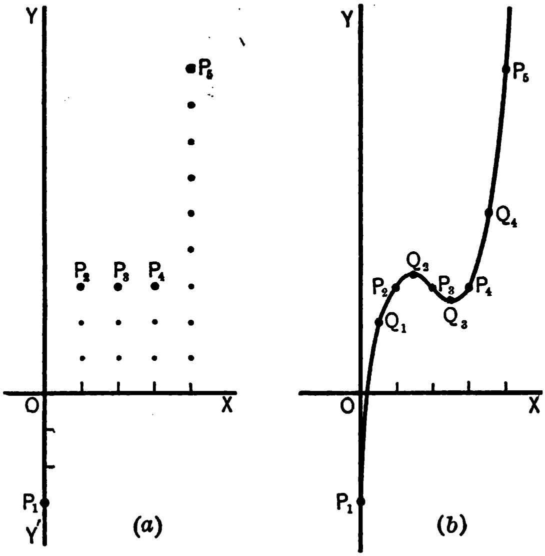


FIG. 16

the additional points $Q_1(0.7, 2.1)$, $Q_2(\frac{3}{2}, \frac{27}{8})$, $Q_3(\frac{5}{2}, \frac{21}{8})$, $Q_4(\frac{7}{2}, \frac{39}{8})$ are plotted in Fig. 16 (*b*), and the curve drawn through them all, the result is as shown in the diagram.

If still greater accuracy is required, especially in that portion of the curve between P_2 and P_4, more values of x must be taken between 1 and 3, and a larger scale must be used. Thus:

$x = 1.2$	1.4	1.75	2.25	2.6	2.8
$y = 3.29$	3.38	3.23	2.77	2.62	2.71
point R_1	R_2	R_3	R_4	R_5	R_6

These with the points P_2, P_3, P_4, Q_2, Q_3 already found are

plotted on a larger scale in Fig. 17 and the corresponding portion

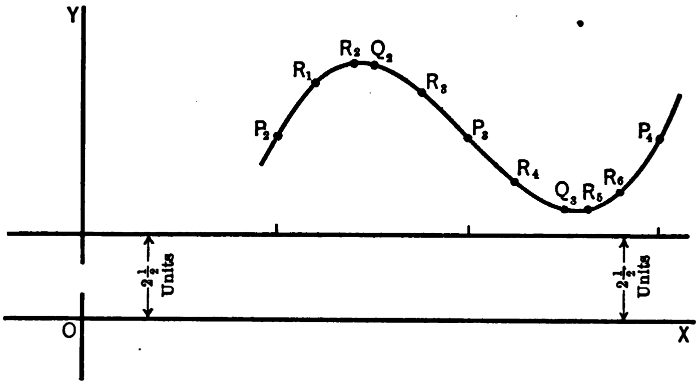


FIG. 17

of the curve sketched through them.

EXERCISES. 1. Construct the graph of each of the following equations

(a) $y = 2x + 3$, (c) $x - 2y - 3 = 0$, (e) $y = x$,
(b) $3y = 4x + 5$, (d) $x - 2y + 3 = 0$, (f) $y = -x$.

2. Construct the graph of each of the following equations

(a) $y = 4x^2 + 7$, (d) $y = x^3$,
(b) $x^2 + 4y - 2x - 3 = 0$, (e) $y = x^3 - 6x^2 + 11x - 6$,
(c) $y^2 = x - 2$, (f) $y = x^4 - x^2 + 2$.

3. Construct on one diagram the graphs of $3x + 4y - 5 = 0$ and $3x + 4y + 2 = 0$. What relation do these graphs bear to each other?

4. Construct on one diagram the graphs of $4y = x^2$ and $4y = x^2 + 8$. In what respect do these two graphs differ?

5. Construct on one diagram the graphs of $4y = x^2$ and $8y = x^2$. In what respect do these two graphs differ?

11. Graphical representation of equations, continued.

The preceding examples show that the construction of the graph

of an equation in two variables x and y consists fundamentally in finding pairs of corresponding values of the variables which satisfy the equation. Points having these coordinates are then marked on the diagram, and the curve or graph is drawn through them. Care must be taken to use enough values to give the curve completely and correctly. The actual labor involved may often be considerably reduced by a careful examination of the form of the equation before any numerical values are substituted in it. The following additional examples will illustrate some of the ways in which this can be done.

1. Construct the graph of $4x^2 + 9y^2 = 36$.

It is often well to begin by examining the equation to see whether the curve passes through the origin, and by finding where it cuts the coordinate axes. In this case the curve does not pass through the origin, because the coordinates (0, 0) do not satisfy the equation.

To find where the curve cuts the X-axis, make $y = 0$ in the equation and solve for x, because at each point where the curve cuts the X-axis the ordinate is zero. Doing this in the equation given above we have $x^2 = 9$, or $x = \pm 3$. Hence the curve cuts the X-axis at two points A, A', three units on either side

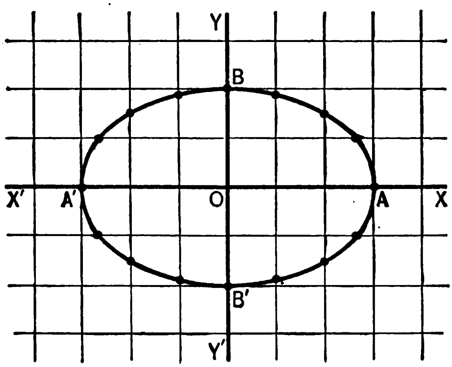


FIG. 18

of the origin. By the same reasoning, making $x = 0$, $y = \pm 2$, and hence the curve cuts the Y-axis at B, B', two units above and below the origin.

If the equation be solved for y,

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}, \qquad (i)$$

from which the following conclusions are drawn:

(*a*) For every value of x there are two equal values of y with opposite signs. Hence the curve is symmetrical with respect to the X-axis.

(*b*) Since x occurs only in the second degree, equal positive and negative values of x lead to the same values of y. Hence the curve is also symmetrical with respect to the Y-axis.

(*c*) If $x^2 > 9$, y is imaginary. Hence the only values of x which make y real lie between -3 and $+3$ and the curve does not extend beyond these limits.

If the equation be solved for x, the result is

$$x = \pm \tfrac{3}{2}\sqrt{4 - y^2}, \qquad (ii)$$

from which, in addition to (*a*), (*b*), (*c*) above, it appears that if $y^2 > 4$, x is imaginary, and hence the curve is included between $y = -2$ and $y = +2$.

To construct the curve some pairs of corresponding values of x and y must be computed. For this purpose either (*i*) or (*ii*) may be used. From the former

$x =$	-3	-2.7	-2	-1	0	$+1$	$+2$	$+2.7$	$+3$
$y =$	0	± 0.87	± 1.49	± 1.89	± 2	± 1.89	± 1.49	± 0.87	0

and the resulting curve is drawn in Fig. 18.

It may be asked what will happen if a value of x greater than 3 or less than -3 be substituted in the equation. It has been shown that the curve does not extend beyond these limits, yet

it is evident that any value we please may be substituted for either x or y. Let $x = 4$ in (i), then the resulting values of y are $\pm \frac{2}{3}\sqrt{-7}$, a pair of imaginary numbers. Hence the corresponding points cannot be constructed. Nevertheless, since the two pairs of coordinates $(4, +\frac{2}{3}\sqrt{-7})$, $(4, -\frac{2}{3}\sqrt{-7})$ both satisfy the equation they are said to be the coordinates of two **imaginary points** on the curve.

2. Construct the graph of $x^2 - y^2 = a^2$.

GENERAL DEDUCTIONS. The curve does not pass through the origin; it cuts the X-axis at $x = +a$, and $x = -a$; it does not cut the Y-axis, because when $x = 0$, y is imaginary.

Since x and y are involved only in the second power the curve is symmetrical with respect to both axes.

Since $y = \pm\sqrt{x^2 - a^2}$, all values of x between $-a$ and $+a$ make y imaginary, hence the curve does not exist in the part of the plane between the two lines parallel to $Y'Y$ at a distance a on either side of it.

Since $x = \pm\sqrt{a^2 + y^2}$, all values of y lead to real values of x. The curve therefore consists of two distinct portions, one lying on the positive side of $x = a$, and the other on the negative side of $x = -a$.

This equation differs in one important respect from those treated up to this point, in that it contains a general constant a^2. Moreover the equation $x^2 - y^2 = a^2$ is homogeneous (See A, I, (c), p. vii) in the three letters x, y, a, and therefore since x and y represent lengths, a is a length also. When an equation contains a general constant, and is homogeneous in x, y and the constant, the best practice in plotting the graph is to assume some arbitrary length for the constant, and then take the values of x or y as multiples of the constant. Thus if $x = 2a$, $y = \pm a\sqrt{3}$, and if $y = \frac{1}{2}a$, $x = \pm\frac{1}{2}a\sqrt{5}$. It is easily seen that with different assumed lengths for the constant the resulting figures will be alike in every respect except size, for the effect

of a change in the length assumed for the constant will be merely to change all linear distances in the figure proportionally.

Taking the equation in the form $x = \pm \sqrt{a^2 + y^2}$, the following table of values is computed.

$y =$	0	$\frac{1}{2}a$	a	$\frac{3}{2}a$	$2a$	$3a$	etc.
$x =$	$\pm a$	$\pm 1.12a$	$\pm 1.41a$	$\pm 1.80a$	$\pm 2.24a$	$\pm 3.16a$	

For negative values of y the resulting values of x are the same as for the corresponding positive values of y. The portion of

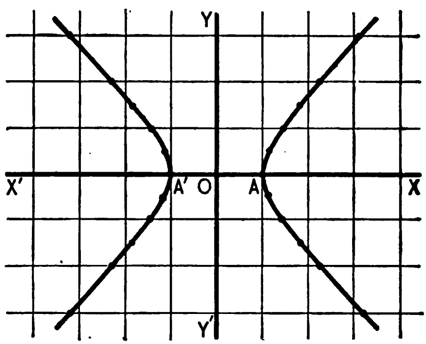


FIG. 19

the curve between $y = -3a$, and $y = +3a$ is drawn in Fig. 19.

3. Plot the graph of

$$y = \frac{3cx}{cx^2 + 2}.$$

This equation like that of example 2 contains a general constant c, but the equation is not homogeneous in the constant and variables. This will perhaps be more clearly seen if the equation be cleared of fractions, $cx^2y + 2y - 3cx = 0$. Hence c is not a length and some numerical value must be assigned to it in order to plot the graph. Let $c = \frac{5}{2}$, then the equation becomes $5x^2y + 4y - 15x = 0$.

Putting $x = 0$ the only resulting value of y is $y = 0$, hence the graph passes through the origin and does not have any other intersection with the Y-axis. Putting $y = 0$, the equation gives $x = 0$, showing that the graph cuts the X-axis only at the origin. It should be noted, however, that as the numerical value of x increases indefinitely, the value of y approaches zero as a limit. To show this solve the equation for y, thus

$$y = \frac{15x}{5x^2 + 4}. \qquad (i)$$

Then dividing numerator and denominator by x

$$y = \frac{15}{5x + \frac{4}{x}}. \qquad (ii)$$

From (ii) it is clear that as the numerical value of x increases indefinitely y approaches zero as a limit. Changing the sign of x, but not its value, changes only the sign of y. Hence the graph is symmetrical with respect to the origin. Using (i) the following table of values for x and y is computed.

$x =$	-5	-4	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0
$y =$	$-\frac{75}{129}$	$-\frac{5}{7}$	$-\frac{45}{49}$	$-\frac{5}{4}$	$-\frac{90}{61}$	$-\frac{5}{3}$	$-\frac{10}{7}$	0 etc.
$=$	-0.58	-0.71	-0.92	-1.25	-1.48	-1.67	-1.43	0

The resulting graph is drawn in Fig. 20.

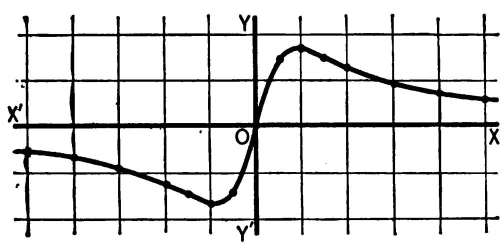


FIG. 20

It will be a valuable exercise to assume different values for c in this equation and construct the corresponding curves, noting the changes thus produced. These will be especially conspicuous if both positive and negative values of c are taken. The student will do well, however, not to consider negative values of c until after he has studied example 4, immediately following.

4. Plot the graph of

$$y = \frac{x(x-2)}{(x+2)(x-1)}.$$

Making $x = 0$, the equation gives $y = 0$, showing that the graph passes through the origin.

Making $y = 0$, the equation gives $x(x-2) = 0$, or $x = 0$ and $x = 2$, showing that the graph cuts the X-axis at $(2, 0)$ as well as at the origin. Thus two points O and A are located on the curve.

For values of x which make one or three of the four factors x, $x - 2$, $x + 2$, $x - 1$, negative, y will be negative, otherwise y will be positive. For all values of $x < -2$ these four factors are all negative, hence y is positive. To the left of $M'M$ therefore the curve lies above $X'X$. If $-2 < x < 0$ (read "x is between -2 and 0") $x + 2$ is positive, but the other three factors are negative, hence y is negative, and the curve lies below $X'X$ between $M'M$ and $Y'Y$. If $0 < x < 1$, y is positive, because two factors x and $x + 2$

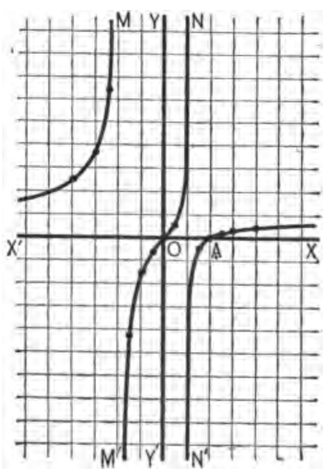


FIG. 21

are positive and the other two negative, hence between $Y'Y$ and $N'N$ the curve lies above OX. Similarly if $1 < x < 2$, y is negative, so that between $N'N$ and A the curve is below OX, and if $x > 2$, y is positive, and hence to the right of A the curve is above OX.

Special attention must be given to values of x in the neighborhood of -2 and $+1$. As x approaches these values, from either side, the corresponding numerical values of y increase without limit, and when $x = -2$ or $+1$ the denominator of the fraction vanishes, and hence the value of y is undefined. This is customarily expressed under such circumstances by writing $y = \infty$.

Without any further information the curve can now be roughly sketched in, but in order to construct it with accuracy the coordinates of a few points must be computed.

$x = -4$	-3	$-2\frac{1}{2}$	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{3}{2}$	$+2\frac{1}{2}$	$+3$	$+4$
$y = \frac{12}{5}$	$\frac{15}{4}$	$\frac{45}{7}$	$-\frac{21}{5}$	$-\frac{3}{2}$	$-\frac{5}{9}$	$\frac{3}{5}$	$-\frac{3}{7}$	$\frac{5}{27}$	$\frac{3}{10}$	$\frac{4}{9}$
$= 2.4$	3.75	6.43	-4.2	-1.5	-0.56	0.6	-0.43	0.18	0.3	0.44

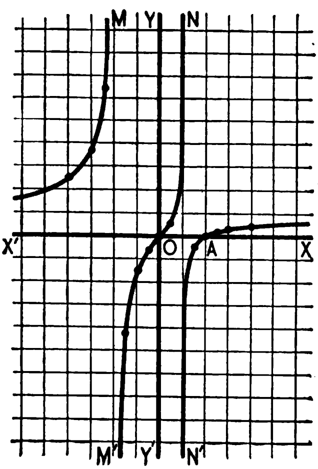


FIG. 21

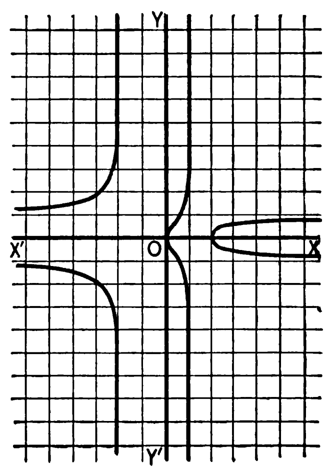


FIG. 22

The part of the curve between these limits is drawn in Fig. 21.

5. Let the student show that the curve in Fig. 22 is the graph of

$$y^2 = \frac{x(x - 2)}{(x + 2)(x - 1)}.$$

12. The fundamental principle of analytic geometry.

From the examples discussed in Arts. 10, 11, it is clear that the graph of an equation has the nature of a locus. It is in fact the locus of points whose coordinates satisfy the equation. Hence the expressions *graph of an equation,* and *locus of an equation* will be used interchangeably in this book.

Definition. *The* ***graph*** *or* ***locus*** *of an equation in two variables is that straight line or curve upon which lies every point whose coordinates are pairs of values of the variables which satisfy the equation; and the coordinates of every point on which satisfy the equation when substituted for the variables.*

This definition is the foundation upon which the whole structure of analytic geometry is built. The measure of the student's apprehension of the principle here set forth will be the measure of his success in mastering the subject.

13. Practical Suggestions.—In following the discussion of the examples in Arts. 10, 11, the student will have seen that the work does not consist exclusively in determining definite corresponding pairs of values of x and y, but that it is desirable by careful examination of the equation to learn as much as possible about the general nature of the graph before undertaking the systematic computation of pairs of values of x and y. Attention was directed in some of the examples to definite methods of procedure by which this can be done. These consist in determining whether the locus passes through the origin, where it cuts the axes, whether it has symmetry with respect to either axis, or with respect to the origin, the range of

values of one variable (if any) which make the other variable imaginary, and the values of one variable (if any) which make the other infinite.

Before undertaking to substitute numerical values for x and y in the equation it is usually best to solve the equation for one of the variables, choosing the one for which this can be done more easily. It will sometimes happen that the equation cannot be solved for either variable. For example, $x^3 + y^3 - 3xy = 0$. Values of x or y may then be substituted in the equation as it stands, and the resulting equation solved for the other variable.

From what has been said it may be concluded that no general rules of procedure can be formulated which apply to all, or even to a majority of cases. The student must depend to a great extent upon his own ingenuity, and precisely for this reason he will find the plotting of loci, or curve tracing as it is called, a valuable exercise, and one which will abundantly repay considerable effort to secure some mastery of it.

EXERCISES. 1. Construct the graph of each of the following equations

(*a*) $xy = 4$, (*b*) $4x^2 + y^2 = 16$, (*c*) $4x^2 - y^2 = -36$,

(*d*) $x^2 + y^2 = 25a^2$, (*e*) $ay^2 = x^3$, (*f*) $xy = c + ax + y$,

(*g*) $y^3 + x^3 = 8$, (*h*) $y^4 + x^4 = 16$, (*i*) $x^3 + y^3 - 4x + 4 = 0$,

(*j*) $y = \dfrac{a^3}{x^2 + a^2}$, (*k*) $y^2 = \dfrac{a^4}{x^2 + 4a^2}$, (*l*) $y^2 = \dfrac{x^2 - 4x}{x + 1}$,

2. Construct on one diagram the graphs of the following equations, and note the differences in the graphs due to the differences in sign in the equations $x^2 + y^2 = 25$, $x^2 - y^2 = 25$, $-x^2 + y^2 = 25$.

3. Construct on one diagram the graphs of the following equations, and note the differences in the graphs due to differences in the values of the numerical coefficients $x^2 + y^2 = 36$, $x^2 + 4y^2 = 36$, $4x^2 + y^2 = 36$, $x^2 + 4y^2 = 25$.

14. Intersection of loci.—Since the graph of an equation is the locus of points whose coordinates satisfy the equation, it

follows that, if two loci are plotted on the same figure, the coordinates of the points where they intersect must satisfy both equations. Hence

RULE. *To determine the coordinates of the points of intersection of two loci solve their equations simultaneously for x and y.*

EXAMPLES. 1. Find the intersection of the loci of $3x - 7y + 2 = 0$ and $x + 2y - 6 = 0$.

Solving the equations for x and y the results are $x = \frac{38}{13}$, $y = \frac{20}{13}$, the coordinates of the required point.

Construct the loci and verify the results from the figure.

2. Determine the intersections of the loci of $x^2 + y^2 = 25$ and $3x + 4y = 25$.

From the second equation $y = \frac{1}{4}(25 - 3x)$, and substituting this value in the first the result is

$$x^2 + \tfrac{1}{16}(25 - 3x)^2 = 25,$$

which reduces to

$$x^2 - 6x + 9 = 0, \qquad \text{or} \qquad (x - 3)^2 = 0,$$

a quadratic with *two equal roots*, $x = 3$.

Substituting $x = 3$ in $x^2 + y^2 = 25$ gives $y = \pm 4$, and making the same substitution in $3x + 4y = 25$ gives $y = 4$ only. The latter is therefore the only value of y which satisfies *both* equations when $x = 3$. Each of the two coincident values $x = 3$ leads therefore to the same value of y, viz., $y = 4$, and the loci are said to meet in two coincident points $(3, 4)$.

Draw the figure and show that the line $3x + 4y = 25$ is tangent to the circle $x^2 + y^2 = 25$ at $(3, 4)$.

15. Solution of equations by intersection of loci.—The approximate solution of a pair of simultaneous equations in two variables can be obtained graphically from the loci of the equations. This method is useful when the algebraic solution cannot be conveniently performed.

For example, let it be required to solve for x and y the equations $x^2 + y^2 - 7x + 2y - 3 = 0$ and $xy + 2x - 4y - 2 = 0$.

The loci of these two equations are drawn in Fig. 23. The points of intersection are P_1 and P_2, whose coordinates are found by measurement to be approximately (2.8, 2.9) and (– 0.4, – 0.6). From the principles of algebra it is evident that there are two other pairs of values of x and y, but since the loci intersect visibly at P_1 and P_2 only, the other solutions are imaginary.

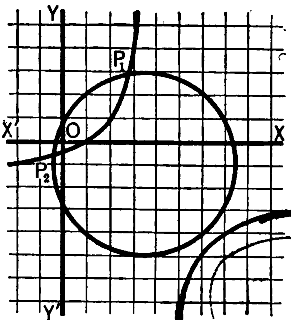


FIG. 23.

16. Imaginaries in analytic geometry. — In several of the preceding discussions of this Chapter reference has been made to imaginary values of x or y. See Ex. 1, p. 21, Ex. 2, p. 22, and the end of Art. 15. These serve to show that imaginary points can exist on real loci, which is only a way of stating in geometric language that in general an equation in two variables can be satisfied by imaginary values of the variables. There are also equations whose loci are wholly imaginary, or which contain at most only a limited number of real points.

DEFINITION. *An equation which is satisfied by only imaginary values of the variables, or by at most a finite number of real values, is called the equation of an* **imaginary locus.**

Three simple types of equations fall under this classification.

(*a*) *Equations one or more of whose coefficients are imaginary numbers.* For example

$$y^2 - (2 + i)x + (3 - 2i)y + 2 = 0. \quad [i = \sqrt{-1}]$$

Putting any real or imaginary value for one variable in this equation and solving for the other variable, the result will in general be imaginary. If, however, the equation be written in the form

$$y^2 - 2x + 3y + 2 - (x + 2y)i = 0$$

it will evidently be satisfied for values of x and y which make

$$y^2 - 2x + 3y + 2 = 0, \qquad \text{and} \qquad x + 2y = 0$$

simultaneously.* Solving these two equations therefore, for x and y, we find the two pairs of real values $x = 7 + \sqrt{41}$, $y = -\frac{7}{2} - \frac{1}{2}\sqrt{41}$, and $x = 7 - \sqrt{41}$, $y = -\frac{7}{2} + \frac{1}{2}\sqrt{41}$, both of which satisfy the original equation, and therefore determine two real points on the imaginary locus. Moreover, these are the only pairs of real values of x and y which can satisfy the given equation.

(b) *Equations all of whose coefficients are real, but which are not satisfied by any real values of the variables.*

Equations of this type consist of, or can be reduced to an even power of a variable term, or the sum of two or more such even powers, equated to a negative constant. For example

$$(x - 2)^2 + y^2 = -3.$$

As the square of a real number is always positive it is clear that an equation of this type cannot be satisfied by any real values of x and y.

(c) *Equations all of whose coefficients are real, but which can be satisfied by only a limited number of pairs of real values of x and y.*

Equations of this type can be reduced to a sum of even powers of variable terms equated to zero. For example $x^2 + y^2 = 0$ is satisfied by the single pair of real values $x = 0$, $y = 0$. Similarly

$$(x^2 - y^2)^2 + (2x - y + 4)^2 = 0$$

is satisfied by those values of x and y which make $x^2 - y^2 = 0$ and $2x - y + 4 = 0$ simultaneously. Solving these two equations for x and y, we find $x = -4$, $y = -4$, and $x = -\frac{4}{3}$, $y = \frac{4}{3}$, which determine the only two real points on the locus.

* See IV (c) p. viii.

EXERCISES. 1. Find from the equations the coordinates of the points of intersection of the graphs of each of the following pairs of equations. Check the results graphically.

(a) $x - 2y + 5 = 0, \quad x^2 + y^2 = 25;$ Ans. (3, 4), (− 5, 0).
(b) $2y^2 = x, \quad 4x^2 + 9y^2 = 25;$ Ans. (2, ± 1).
(c) $16y - 15x + 99 = 0, \quad x^2 - 4y^2 = 16;$

2. Show by finding the coordinates of their points of intersection that the graphs of the following pairs of equations are tangent to each other. Check graphically.

(a) $x^2 + y^2 = 25, \quad 3x + 4y - 25 = 0;$
(b) $4x^2 + 9y^2 = 25, \quad 8x + 9y + 25 = 0;$
(c) $xy = 4, \quad x + y - 4 = 0;$

3. Find approximate values graphically for the coordinates of the points of intersection of the graphs of the following pairs of equations

(a) $x^2 - 4y^2 = 4, \quad y = \dfrac{8}{4 + x^2};$ (c) $y^2 = \dfrac{x(x-1)}{x^2 - 4}, \quad 8y = x^3.$
(b) $x^2 + y^2 = 25, \quad y^2 - x^3 = 0;$

4. Find the coordinates of the real points in each of the graphs whose equations are

(a) $y^2 + (3 + i)x + (7i - 4)y - 3 - 23i = 0;$
Ans. (2, 3), (− 131, 22).
(b) $(2 + i)x + (3i - 4)y + 10 - 5i = 0;$ Ans. (− 1, 2).
(c) $(1 + i)x^2 + (1 - i)y^2 - 13 + 5i = 0.$ Ans. (± 2, ± 3).

5. Show that there is but one real point on each of the graphs whose equations are

(a) $(3 + \sqrt{-4})x + (5\sqrt{-1} - 4)y - 6 + 19\sqrt{-1} = 0;$
(b) $x^2 + i(y^2 + x^2) + 2ixy - 4x + 4 = 0;$
(c) $x^2 + y^2 - 4x + 6y + 13 = 0.$

6. Show that there is no real point on the graph of any of the following equations

(a) $2x^2 + y^2 + ix - 2(2 + i)y + 40 - 4i = 0;$
(b) $x^2 + y^2 + 4 = 0;$ (c) $x^2 + 4y^2 - 2x + 5 = 0.$

7. Show graphically that the roots of the following pairs of equations are imaginary

(a) $x^2 + y^2 = 4, \qquad x + y = 4;$
(b) $y^2 - 4x + 4 = 0, \qquad y^2 + 4x + 4 = 0;$

17. Functions.—The notation $f(x, y)$ is often used to denote in general an expression containing the variables x and y. It is read: "function of x and y," or "f of x and y." Hence $f(x, y) = 0$ is a short way of representing in algebraic language any equation in the two variables x and y all the terms of which have been transposed to the first member. In a specific problem or discussion $f(x, y)$ may stand for an expression of definite form, and when so used it represents the same form of expression throughout the discussion. Thus in one discussion we may have $f(x, y) \equiv x^2 + y^2 - a^2$, in another $f(x, y) \equiv 3x + 2y - 6$, etc.

If $f(x, y) \equiv x^2 - 3y^2 + 6x - 4$, then in the same discussion $f(a, b)$ means the expression obtained by substituting a for x, and b for y, hence $f(a, b) \equiv a^2 - 3b^2 + 6a - 4$. Similarly
$f(2, 3) \equiv 2^2 - 3(3)^2 + 6(2) - 4 = 4 - 27 + 12 - 4 = -15,$
$f(1, 0) \equiv 1^2 - 3(0)^2 + 6(1) - 4 = 3, \qquad f(0, 0) = -4$, etc.

18. Composite loci.

THEOREM. *If the expression $f(x, y)$ is factorable, the locus of the equation $f(x, y) = 0$ consists of as many distinct lines and curves as there are variable factors of $f(x, y)$.*

Let $f(x, y)$ consist of three factors, $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y)$. Then the equation $f(x, y) = 0$ can be written in the form

$$f_1(x, y) \cdot f_2(x, y) \cdot f_3(x, y) = 0. \qquad (i)$$

Any values of x and y which satisfy the equation

$$f_1(x, y) = 0$$

will also satisfy equation (i), and similarly with reference to $f_2(x, y) = 0$ and $f_3(x, y) = 0$. Hence all points on the three

separate loci of

$$f_1(x, y) = 0, \qquad f_2(x, y) = 0, \qquad f_3(x, y) = 0$$

will also be points on the locus of $f(x, y) = 0$.

The same argument holds no matter what number of factors $f(x, y)$ may have. The theorem is therefore proved.

If $f(x, y)$ has equal factors, the corresponding component of the complete locus is counted as many times as the factor occurs.

DEFINITION. *A locus which is thus made up of separate parts is called a* **composite locus.**

19. Equations containing only one variable.—Equations which contain only one variable explicitly, for example $2x - 3 = 0$, may always be interpreted as containing the other variable also with coefficient zero. For all values of y in the equation $2x - 3 = 0$, or $2x + 0y - 3 = 0$, x has the same value, $x = \frac{3}{2}$. Hence the locus consists of a straight line parallel to the Y-axis and $1\frac{1}{2}$ units on the positive side, because this line contains all points having the abscissa $\frac{3}{2}$, and any ordinate whatsoever.

Similarly $y^2 - 5y - 6 = 0$, or $(y - 6)(y + 1) = 0$ is the equation of the two lines parallel to the X-axis, one six units above and the other one unit below this axis.

EXERCISES. 1. Construct the complete locus of each of the following equations

(a) $x^2 = y^2$,

(b) $x^2 - 4xy + 4y^2 - 4 = 0$,

(c) $x^3 + xy^2 - 4x = 0$,

(d) $x^4 - y^4 = 0$,

(e) $xy - 3x + 5y - 15 = 0$,

(f) $x^3 - 6x^2 + 11x - 6 = 0$.

2. Show that the locus of the equation $Ax^2 + Bx + C = 0$ is a pair of parallel lines, a pair of coincident lines, or a pair of imaginary lines, according as $B^2 - 4AC$ is positive, zero, or negative.

4. What is the form of the equation of a line parallel to the X-axis? the Y-axis?

5. What is the equation of the X-axis? the Y-axis?

6. Find for each of the following sets of equations a single equation

whose locus will be the combination of the loci of the separate equations.

(a) $x = y$, $x = -y$. Ans. $x^2 - y^2 = 0$.

(b) $x = 0$, $x = 4$, $x = 6$. Ans. $x^3 - 10x^2 + 24x = 0$.

(c) $x = y$, $x^2 + y^2 = 4$, (d) $x^2 + y^2 = 8$, $xy + 4 = 0$.

20. Determination of the equation of a given locus.— So far in this chapter we have discussed the construction of the loci of given equations. The inverse problem of finding the equation from the locus, or from its definition, is a vital part of the subject.

EXAMPLES. 1. A point moves in a plane so as to be always equally distant from the points (2, 6), (4, − 2). Find the equation of the locus of the moving point.

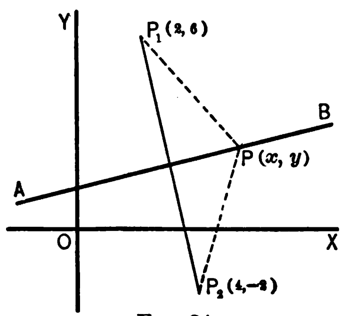


FIG. 24

Let $P_1 = (2, 6)$, and $P_2 = (4, -2)$, then we know from elementary geometry that the required locus is AB, the perpendicular bisector of P_1P_2. To determine the equation of AB take any representative point $P(x, y)$ on it. Then by (3), p. 6, the distance

$$PP_1 = \sqrt{(x-2)^2 + (y-6)^2},$$

and

$$PP_2 = \sqrt{(x-4)^2 + (y+2)^2}.$$

By the conditions stated these are equal, hence

$$\sqrt{(x-2)^2 + (y-6)^2} = \sqrt{(x-4)^2 + (y+2)^2} \qquad (i)$$

which is the required equation.

Equation (i) may be simplified. Thus squaring and expanding, it becomes

$$x^2 - 4x + 4 + y^2 - 12y + 36 = x^2 - 8x + 16 + y^2 + 4y + 4,$$

which reduces to $\qquad x - 4y + 5 = 0.$

2. A point moves in a plane so that the sum of the squares of its distances from two fixed points in the plane is constant. What locus will it describe?

The problem is stated without reference to any particular lines as axes of coordinates. Any two lines in the plane may be used for this purpose. The *form* of the resulting equation will depend upon the choice of axes, but the *geometric properties* expressed by the equation will be the same in all cases. It is best to choose the axes so that the resulting equation will be as simple as possible.

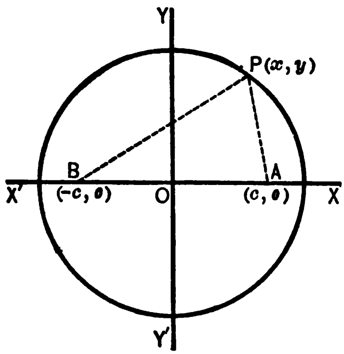


FIG. 25

In this problem take the line joining the two given points as X-axis, with the origin half way between them. Hence let $A(c, 0)$ and $B(-c, 0)$ be the two given points, and $P(x, y)$ any point on the required locus. Then since it is given that $\overline{AP}^2 + \overline{BP}^2 = k^2$, where k^2 is a constant, we have by (3) p. 6

$$(x - c)^2 + y^2 + (x + c)^2 + y^2 = k^2,$$

which reduces to $\qquad x^2 + y^2 = \frac{1}{2}k^2 - c^2.$

If this be plotted for definite values of k^2 and c it will be found that the locus is a circle with center half way between the given points.

EXERCISES. In each of the following exercises find the equation of the locus of a point which moves in a plane according to the conditions stated:

1. So as to be always equally distant from the origin and the point (2, 4). Ans. $x + 2y = 5$.

2. So that the sum of its distances from $(-2, 0)$ and $(2, 0)$ is always equal to 6. Ans. $5x^2 + 9y^2 = 45$.

3. So that its distance from the origin is always equal to the slope of the line joining it to the origin. Ans. $y^2(1 - x^2) = x^4$.

4. So that its distance from a fixed point is twice its distance from a fixed line.

5. So that its distance from the point $(2, 4)$ is twice its distance from the point $(-1, 4)$. Ans. $x^2 + y^2 + 4x - 8y + 16 = 0$.

EXERCISES ON CHAPTER II

Normal Exercises

1. Construct the graph of each of the following equations

(a) $x^2 - y^2 = 4a^2$,
(b) $y^2 = x(x^2 - 9)$,
(c) $y = \dfrac{x(x-4)}{x^2 - 4}$,
(d) $y = ax + 4$,
(e) $x^2y^2 - 4 = 0$,
(f) $x^2 - 6x + 5 = 0$,
(g) $y = 4$.

2. Find from the equations the coordinates of the points of intersection of the graphs of each of the following pairs of equations

(a) $x^2 + y^2 = 10, \quad x^2 + y^2 - 10x = 0$; Ans. $(1, \pm 3)$.
(b) $x^2 - y^2 = 0, \quad y + 3x - 6 = 0$. Ans. $(3, -3)$, $(\frac{3}{2}, \frac{3}{2})$.

3. Find from their graphs approximate values of the roots of the following pairs of simultaneous equations

(a) $y(x^2 + 5) = 8, \quad y = x^4 - 4x^2 + 2$;
(b) $x^2y + xy^2 = 4, \quad 8x^3 - y^3 = 0$.

4. Determine for each of the following pairs of equations the coordinates of the points of intersection of their graphs and hence show that the graphs are tangent to each other

(a) $y^2 = 4x, \quad 2x + 4y + 8 = 0$;
(b) $x^2 - y^2 = 4, \quad x^2 + y^2 + 2x - 8 = 0$.

5. Find the coordinates of the real points on the graphs of each of the following equations

(a) $(2 + 3i)x + (4 - i)y - 6i = 0$, Ans. $(\frac{12}{7}, -\frac{6}{7})$.
(b) $(1 + i)x + (2 - 3i)y - 6 - i = 0$, Ans. $(4, 1)$.
(c) $x^2 + 2y^2 - 4x - 12y + 22 = 0$, Ans. $(2, 3)$.
(d) $4x^2 + y^2 + 4y + 4 = 0$. Ans. $(0, -2)$.

6. Show that there are no real points on the graph of any of the following equations

(a) $x^2 + y^2 + 4 = 0$, (b) $2x^2 + 3y^2 + 4x - 12y + 16 = 0$,
(c) $(4 + i)x^2 + 3y^2 + (3 + 4i)x + 4 + 5i = 0$,
(d) $x^2 + 2y^2 + ix - 3iy + 6 - 7i = 0$.

7. A point moves in a plane so that the ratio of its distances from the points (2, 4) and (− 3, 7) is always as 1 to 3. Find the equation of its locus. Ans. $4x^2 + 4y^2 - 21x - 29y + 61 = 0$.

8. Find the equation of the locus of a point which moves in a plane so that the slope of the line joining it to the point (2, − 4) is always equal to the slope of the line joining it to the point (3, − 5).

Ans. $x + y + 2 = 0$.

General Exercises

9. Construct the graphs of the following equations

(a) $x = 2$, (f) $x^2 - 4y^2 = 36$, (k) $x^3 - y = 0$,
(b) $x = 2y$, (g) $x^2 - 4y = 36$, (l) $yx = a^2$,
(c) $x = 2y + 2$, (h) $x^2 - 4y^2 = 0$, (m) $yx^2 = a^3$,
(d) $x^2 + y^2 = 36$, (i) $x^2 - 4 = 0$, (n) $y^2(x^2 - 4) = x^4$.
(e) $x^2 + 4y^2 = 36$, (j) $y^2(x^2 - 3x + 2) = x(x + 4)$,

10. Construct on one diagram the graphs of the following equations

$$x + y = 2, \quad x^2 + y^2 = 4, \quad x^3 + y^3 = 8, \quad x^4 + y^4 = 16.$$

11. Construct on one diagram the graphs of the following equations

$$y = 4, \quad y + \tfrac{1}{2}x = 4, \quad y + x = 4, \quad y + 2x = 4.$$

12. Construct on one diagram the graphs of the equation $y = x + b$, for b equal to 0, 2, 4, and − 2 respectively. In what respect do these graphs differ from each other?

13. Construct on one diagram the graphs of the equation $xy^n = 1$, for n equal to − 2, − 1, − ½, 0, ½, 1, and 2 respectively. What is common to all these graphs? What properties are common to the graphs with negative values of n? What for positive values of n?

14. Construct the graph of the equation $y^2 = (x - a)(x - 1)(x - 4)$ (i) for $a = -1$, (ii) for $a = 0$, (iii) for $a = +1$.

15. Find the value of A so that there will be only one real point on the graph of the equation $x^2 - 4x + 4y^2 - 16y + A = 0$. Ans. $A = 20$.

16. For what range of values of A will the graph of the equation of exercise 15 be entirely imaginary? Ans. $A > 20$.

17. Find the coordinates of the real points of the graph of each of the following equations

(a) $x^2 + y^2 - 4x + 4 = 0$, Ans. $(2, 0)$.

(b) $(1 + i)x + (1 - i)y = 10$, Ans. $(5, 5)$.

(c) $(1 + 2i)x^2 + (8i - 4)xy + (4 + 2i)y^2 = 0$, Ans. $(0, 0)$.

18. Determine which of the graphs of the following equations are tangent to each other $5x - 4y = 9$, $x^2 - y^2 = 9$, $x^2 + y^2 = 9$, $x^2 + y^2 - 12x + 10y + 20 = 0$.

19. A point moves so that the difference of its distances from the points $(2, -4)$ and $(2, 4)$ is always equal to 6. Determine the equation of its locus, and construct the locus from the equation.

Ans. $9x^2 - 7y^2 - 36x + 99 = 0$.

20. A point moves so that the slope of the line joining it to the point $(0, 2)$ is twice the slope of the line joining it to the point $(0, -2)$. Determine the equation of its locus, and construct the locus from the equation. Ans. $y = -6$.

21. $A = (2, 4)$ and $B = (-1, 3)$ are fixed points. The variable point P moves so that the area of the triangle ABP is always equal to 6. Find the equation of its locus, and construct the locus from the equation.

Ans. $x - 3y - 2 = 0$, and $x - 3y + 22 = 0$.

22. A point moves so that the slope of the line joining it to the point $(2, 4)$ is always equal to the square of its distance from the Y-axis. Find the equation of its locus and construct the locus from the equation.

Ans. $y = x^3 - 2x^2 + 4$.

23. Let $A = (0, -2)$, and let AP, any line through A, cut the X-axis at M. If P moves so that MP always equals 3, find the equation of the locus of P, and construct the locus.

Ans. $x^2y^2 + (y^2 - 9)(y + 2)^2 = 0$.

24. A point moves so that its distance from the origin is always equal to four times the cosine of the angle which the line joining it to the origin makes with the positive extension of the X-axis. Find the equation of its locus, and construct the locus.

Ans. $x^2 + y^2 - 4x = 0$.

CHAPTER III

THE STRAIGHT LINE

21. Conditions which determine a straight line.—In plane geometry a straight line is said to be *determined* when its position in the plane is definitely fixed. Elementary geometry teaches that a straight line is thus determined by *two conditions*, properly chosen. For example, a straight line is determined when two points on it are given. In this chapter several standard forms of the equation of the straight line, determined by prescribed sets of conditions, will be derived.

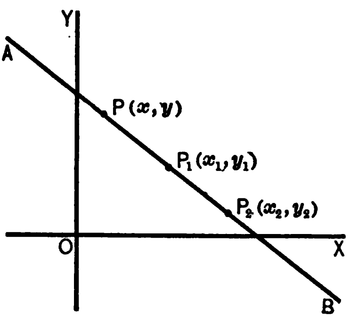


FIG. 26

22. Problem.— *To derive the equation of a straight line in terms of the coordinates of two given points on the line.*

Let AB be the straight line determined by the two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and let $P(x, y)$ be a representative point on AB. Since P, P_1, P_2 are all on the same line AB, the slopes of PP_1 and of P_1P_2 are equal. Hence by (4), p. 7, the required equation is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}, \tag{1}$$

which may also be written in the form

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \tag{2}$$

Either (1) or (2) is known as the **two-point form** of the equation of the straight line.

If one of the given points, say (x_1, y_1), is the origin $(0, 0)$, equation (2) takes the form

$$y = \frac{y_2}{x_2} x, \tag{3}$$

which is the equation of the straight line through the origin and another given point.

The student familiar with the determinant notation can verify the fact that equation (1) can be written in the form

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = 0. \tag{4}$$

23. Problem.—*To derive the equation of a straight line in terms of the slope and the coordinates of a given point on the line.*

In equation (2), Art. 22, the factor $\dfrac{y_1 - y_2}{x_1 - x_2}$ on the right is the slope m, see (4), p. 7. Hence we may write

$$y - y_1 = m(x - x_1), \tag{5}$$

which is the required equation. This is known as the **slope and one-point form** of the equation of the straight line.

If the given point is the origin, $x_1 = 0$, $y_1 = 0$, and hence (5) becomes

$$y = mx, \tag{6}$$

which is the equation of the line through the origin having the slope m. Compare this result with (3) and show by a figure that they are consistent.

DEFINITION. *The distances from the origin to the points where a line cuts the coordinate axes are called the* ***intercepts.***

The x-intercept, OA, Fig. 27, is designated by a, and the y-intercept, OB, by b. The signs of the intercepts are positive

or negative according to the location of the intersections A and B with respect to the origin. In the figure the intercept $a = OA$ is negative, and the intercept $b = OB$ is positive.

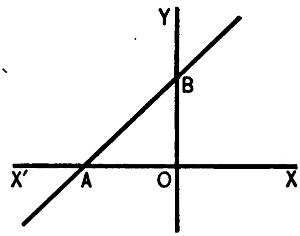


FIG. 27

24. Problem. — *To derive the equation of a straight line in terms of the slope and y-intercept.*

Having given the y-intercept, OB, Fig. 27, is equivalent to having given the point $(0, b)$ on the line, hence the result can be obtained from (5) by substituting $x_1 = 0$, $y_1 = b$ in that equation. This gives

$$y - b = mx,$$

or

$$\boldsymbol{y = mx + b}. \tag{7}$$

This is known as the **slope and y-intercept form** of the equation of the straight line.

25. Problem.—*To derive the equation of a straight line in terms of its intercepts.*

The coordinates of A and B, Fig. 27, where the line cuts the X- and Y-axes are respectively $(a, 0)$ and $(0, b)$. Substituting these for (x_1, y_1) and (x_2, y_2) respectively in equation (1) we have

$$\frac{y}{x - a} = \frac{-b}{a},$$

which reduces to $$bx + ay = ab,$$

or dividing by ab $$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1}. \tag{8}$$

This is known as the **intercept form** of the equation of the straight line.

26. Problem.—*To derive the equation of the straight line in*

terms of the length of the perpendicular from the origin to the line and the angle which this perpendicular makes with OX.

The length OD of the perpendicular from the origin to any line AB is designated by p, and the angle XOD by α. The length p is taken as positive for all positions of AB, and α is always measured around from OX to OD, so that, according to the position of AB, α may have any value from 0° to 360°.

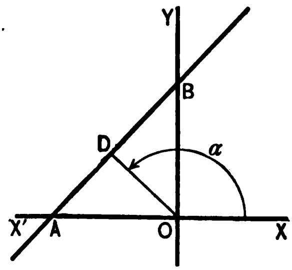


FIG. 28

To derive the equation of AB in terms of p and α we shall use equation (5), Art. 23, because both the slope m, and the coordinates of D, a point on the line, can easily be expressed in terms of p and α. Thus

$$m = \tan OAD = \cot DOA = -\cot \alpha, \tag{i}$$

and if $D = (x_1, y_1)$ then for all positions of AB

$$x_1 = p \cos \alpha, \qquad y_1 = p \sin \alpha. \tag{ii}$$

Substituting the values from (*i*) and (*ii*) in (5), we have

$$y - p \sin \alpha = -\cot \alpha(x - p \cos \alpha).$$

Multiplying by $\sin \alpha$ this becomes

$$y \sin \alpha - p \sin^2 \alpha = -\cos \alpha(x - p \cos \alpha),$$

$$= -x \cos \alpha + p \cos^2 \alpha.$$

$$\therefore \qquad x \cos \alpha + y \sin \alpha - p(\sin^2 \alpha + \cos^2 \alpha) = 0$$

or

$$\mathbf{x \cos \alpha + y \sin \alpha - p = 0,} \tag{9}$$

which is the required equation.

This is known as the **normal form** of the equation of the straight line.

27. Recapitulation.—For convenience of reference the principal forms of the equation of the straight line are here repeated.

Two-point form $\quad y - y_1 = \dfrac{y_1 - y_2}{x_1 - x_2}(x - x_1).$ (2)

Slope and one-point form $\quad y - y_1 = m(x - x_1).$ (5)

Slope and y-intercept form $\quad y = mx + b.$ (7)

Intercept form $\quad \dfrac{x}{a} + \dfrac{y}{b} = 1.$ (8)

Normal form $\quad x \cos \alpha + y \sin \alpha - p = 0.$ (9)

EXERCISES. In each part of exercises 1 to 4 first construct the line determined by the given quantities. Then write its equation, using the appropriate formula from those given above. Finally reconstruct the line from the equation, and see whether it coincides with the figure first drawn.

1. (*a*) Two points $(2, 3)$, $(-5, -1)$. Ans. $4x - 7y + 13 = 0$.
   (*b*) $a = -3$, $b = 2$. (*c*) $m = -2$, $b = 1$.
   (*d*) $m = \frac{2}{3}$, one point $(2, -5)$. (*e*) $p = 3$, $\alpha = 60°$.
2. (*a*) $m = -\frac{4}{3}$, one point $(-2, -1)$. (*b*) $p = 3$, $\alpha = 210°$.
   (*c*) $m = -\frac{1}{2}$, $b = -2$. Ans. $x + 2y + 4 = 0$.
   (*d*) Two points $(-6, 2)$, $(3, -4)$. (*e*) $a = 2$, $b = 3$.
3. (*a*) $a = 6$, $b = 5$. (*b*) $m = -1$, one point $(3, 0)$.
   (*c*) $p = 3$, $\alpha = 135°$. Ans. $x - y + 3\sqrt{2} = 0$.
   (*d*) Two points $(0, 0)$, $(-4, 7)$. (*e*) $m = \frac{5}{2}$, $b = -3$.
4. (*a*) $m = 1$, $b = 3$. (*b*) Two points $(5, 3)$, $(2, -3)$.
   (*c*) $p = 3$, $\alpha = 315°$. (*d*) $m = \frac{5}{3}$, one point $(0, 0)$.
   (*e*) $a = -3$, $b = -5$. Ans. (*b*) $5x + 3y + 15 = 0$.

5. Construct the line for which $p = 2$, and $\alpha = 0°$. What are the values of m, a and b for this line? Ans. $m = \infty$, $a = 2$, $b = \infty$.

6. Construct the line for which $p = 2$, and $\alpha = 180°$. What are the values of m, a and b for this line? Ans. $m = \infty$, $a = -2$, $b = \infty$.

7. Construct the line for which $p = \sqrt{2}$, and $\alpha = 225°$. What are the values of m, a and b for this line? Ans. $m = -1$, $a = b = -2$.

8. Derive the slope and y-intercept form of the equation of a straight line directly from a figure.

9. Derive the intercept form of the equation of a straight line directly from a figure.

28. The parameters of a straight line.

DEFINITION. *The constants m, a, b, p, α used in equations* (5), (7), (8) *and* (9) *are called* ***parameters*** *or* ***determining constants*** *of the straight line.*

For every straight line these parameters have definite values. The following equations express some of the relations connecting them.

I. $m = -\cot\alpha$. For proof see p. 43, equation (i).

II. $m = -\dfrac{b}{a}$. This is proved by Fig. 27, because

$$m = \frac{OB}{AO} = -\frac{b}{a},$$

since $OB = b$, and $AO = -OA = -a$.

III. $\tan\alpha = \dfrac{a}{b}$. This follows from I and II.

These relations between the parameters serve to emphasize the fact, otherwise evident, that the five parameters of a straight line are not independent quantities. In fact arbitrary values can never be assigned to more than two of them.

29. The straight line referred to oblique axes.

Let AB, Fig. 29, be a straight line determined by the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ referred to oblique axes, and let P be a representative point on AB. Draw MP, M_1P_1, M_2P_2 parallel to OY, and P_1K, P_2K_1 parallel to OX. Then the triangles P_1KP and $P_2K_1P_1$ are similar, and

$$\frac{KP}{P_1K} = \frac{K_1P_1}{P_2K_1}. \qquad (i)$$

But

$$KP = y - y_1, \qquad P_1K = x - x_1, \qquad (ii)$$

$$K_1P_1 = y_1 - y_2, \qquad P_2K_1 = x_1 - x_2. \qquad (iii)$$

Hence, substituting from (*ii*) and (*iii*) in (*i*),

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2},$$

from which it is seen that the two-point form of the equation of the straight line is the same whether the axes are oblique or

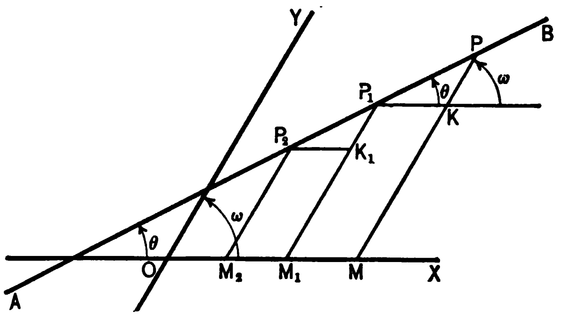


FIG. 29

rectangular. The same therefore is true of the intercept form, which is only a special case of the other. In other words equations (1), (2), (3), (4), Art. 22, and (8), Art. 25, can be used with both rectangular and oblique axes.

The same is not true with regard to those forms of the equation of the straight line which involve the slope. Let ω be the angle between the axes and θ the angle which AB makes with OX. Then from the triangle P_1KP

$$\frac{KP}{P_1K} = \frac{\sin KP_1P}{\sin P_1PK}, \qquad (iv)$$

where

$$KP_1P = \theta, \qquad \text{and} \qquad P_1PK = \omega - \theta, \qquad (v)$$

hence, substituting from (*ii*) and (*v*) in (*iv*),

$$\frac{y - y_1}{x - x_1} = \frac{\sin \theta}{\sin (\omega - \theta)},$$

or

$$y - y_1 = \frac{\sin \theta}{\sin (\omega - \theta)} (x - x_1), \tag{10}$$

which is the form of the equation corresponding to (5) when the axes are oblique.

Similarly

$$y = nx + b, \qquad \text{where} \qquad n = \frac{\sin \theta}{\sin (\omega - \theta)}, \tag{11}$$

will be the form corresponding to (7).

30. Two fundamental theorems.

I. *To every straight line corresponds an equation of the first degree in two variables.*

This has been proved in the foregoing discussion. Each form of the equation of the straight line has been found to be of the first degree in two variables.

II. *Conversely, every equation of the first degree in two variables is the equation of a straight line.*

Let $Ax + By + C = 0$ be any equation of the first degree in two variables, where A, B, C are constants which can have any finite value, including zero, except that A and B cannot both be zero.

If $B \neq 0$ the equation can be transformed into

$$y = -\frac{A}{B}x - \frac{C}{B}$$

by transposing and dividing by B. It is now in the form $y = mx + b$, hence it is the equation of the straight line whose slope m is $-(A/B)$, and whose y-intercept b is $-(C/B)$.

If $B = 0$, the equation takes the form $Ax + C = 0$, or $x = -(C/A)$, which is the equation of a straight line parallel to the y-axis. See Art. 19.

31. Reduction of the general equation $Ax + By + C = 0$ to the forms (7), (8), (9).

It has been shown immediately above how to reduce $Ax + By + C = 0$ to the form (7), $y = mx + b$.

Similarly to reduce the same equation to the form (8),

$$\frac{x}{a} + \frac{y}{b} = 1,$$

transpose the absolute term to the right, and divide by $-C$.

This gives
$$\frac{Ax}{-C} + \frac{By}{-C} = 1.$$

Then dividing numerator and denominator of the two fractions by A and B respectively,

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1,$$

which is in the required form.

EXAMPLE. The equation $3x - 2y + 8 = 0$ when solved for y becomes $y = \frac{3}{2}x + 4$. It is now in the form $y = mx + b$, so that $m = \frac{3}{2}$, $b = 4$.

To reduce the same equation to the form (8) we have successively

$$3x - 2y = -8, \qquad -\tfrac{3}{8}x + \tfrac{1}{4}y = 1,$$

and
$$\frac{x}{-\frac{8}{3}} + \frac{y}{4} = 1,$$
so that $a = -\frac{8}{3}$, $b = 4$.

THEOREM. *To reduce the equation* $Ax + By + C = 0$ *to the form* (9), $x \cos \alpha + y \sin \alpha - p = 0$, *divide by* $+\sqrt{A^2 + B^2}$ *if* C *is negative, or by* $-\sqrt{A^2 + B^2}$ *if* C *is positive.*

Trigonometry teaches that two numbers are the sine and cosine of an angle when, and only when, the sum of their squares is unity. Hence to reduce $Ax + By + C = 0$ to the form $x \cos \alpha + y \sin \alpha - p = 0$ it must be divided by a number such that the sum of the squares of the resulting coefficients of x and

y shall be unity. Let k be this number. Then

$$\frac{A}{k}x + \frac{B}{k}y + \frac{C}{k} = 0, \tag{i}$$

where k is to be chosen so that

$$\left(\frac{A}{k}\right)^2 + \left(\frac{B}{k}\right)^2 = 1. \tag{ii}$$

Solving (ii) for k it is found that

$$k = \pm\sqrt{A^2 + B^2}. \tag{iii}$$

Hence, substituting from (iii) in (i),

$$\frac{A}{\pm\sqrt{A^2 + B^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2}}y + \frac{C}{\pm\sqrt{A^2 + B^2}} = 0, \tag{iv}$$

which is in the form $x \cos \alpha + y \sin \alpha - p = 0$, provided the sign of the radical $\sqrt{A^2 + B^2}$ is chosen so as to make $\dfrac{C}{\pm\sqrt{A^2 + B^2}}$ a negative number. That is, if C is positive, $\sqrt{A^2 + B^2}$ must be taken with the negative sign, and *vice versa*. Thus the theorem is proved.

The student should make careful note of this theorem, as the normal form of the equation of the straight line is the form most conveniently used in solving an important class of problems.

Exercises. 1. Construct the locus of each of the following equations. Reduce each equation in succession to the forms of equations (7), (8), (9). Determine the values of m, a, b, p, α, and verify the results so obtained by reference to the figure.

(a) $2x - y - 5 = 0,$	(f) $3y - 4x - 5 = 0,$
(b) $3x + 4y - 15 = 0,$	(g) $y + 2x + 7 = 0,$
(c) $x - 2y + 6 = 0,$	(h) $5y - 5x + 12 = 0,$
(d) $5x + 12y + 13 = 0,$	(i) $5y - 12x - 13 = 0,$
(e) $6x - 5y + 2 = 0,$	(j) $y + 3x + 2 = 0.$

Ans. (a) $y = 2x - 5,\quad \dfrac{x}{\frac{5}{2}} + \dfrac{y}{-5} = 1,\quad \dfrac{2x - y - 5}{\sqrt{5}} = 0,$

$m = 2,\quad a = \frac{5}{2},\quad b = -5,\quad p = \sqrt{5},\quad \sin \alpha = -\frac{1}{5}\sqrt{5},\quad \cos \alpha = \frac{2}{5}\sqrt{5}.$

2. Show that $y = m(x - a)$ is the equation of a straight line in terms of the parameters m and a.

3. The vertices of a triangle are (2, 7), (5, − 3), (− 3, 2), find the equations of the three sides.

Ans. $10x + 3y - 41 = 0$, $x - y + 5 = 0$, $5x + 8y - 1 = 0$.

4. The equations of the three sides of a triangle are $7x - 4y - 1 = 0$, $8x + 3y - 39 = 0$, $x + 7y + 15 = 0$, find the coordinates of the vertices, and the equations of the medians.

Ans. (3, 5), (− 1, − 2), (6, − 3), $15x - y - 40 = 0$, $6x - 11y - 16 = 0$, $9x + 10y - 24 = 0$.

5. What series of straight lines is represented by equation (8) if a remains fixed in value, and b takes all possible values?

6. If in equation (9) p remains constant and α takes all possible values what series of lines will be produced?

7. What relation will hold between the coefficients A, B, C of the equation $Ax + By + C = 0$, if

(*a*) the x- and y-intercepts are equal? Ans. $A = B$.

(*b*) the line makes an angle of 45° with OX? Ans. $A = -B$.

(*c*) the line is tangent to the circle of radius r with its center at the origin? Ans. $C^2 = r^2(A^2 + B^2)$.

(*d*) the line passes through the origin?

(*e*) the line passes through the point (− 1, 1)?

(*f*) the line makes an angle of 135° with OX?

(*g*) the x-intercept $= 2p$?

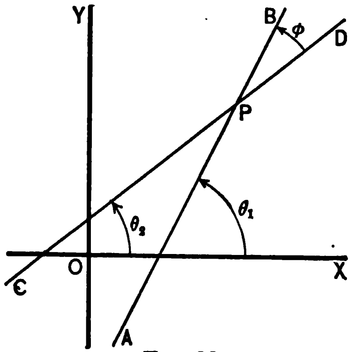


FIG. 30

32. The angle between two straight lines.

DEFINITION. *The angle which a straight line makes with a second line is the angle through which the second line must be turned, in the positive direction of rotation, to bring it into coincidence with the first line.*

According to the definition

the angle which AB makes with CD is the angle DPB (or the angle CPA), but the angle which CD makes with AB is the angle BPC (or APD).

Theorem. *If ϕ be the angle which line AB, with slope m_1, makes with line CD, with slope m_2, then*

$$\tan \phi = \frac{m_1 - m_2}{1 + m_1 m_2}. \tag{12}$$

Let $y = m_1 x + b_1$ be the equation of AB, and $y = m_2 x + b_2$ that of CD, and let the angles which these lines make with $X'X$ be θ_1 and θ_2 respectively. Then $m_1 = \tan \theta_1$, and $m_2 = \tan \theta_2$.

From the figure $\phi = CPA = \theta_1 - \theta_2$, hence

$$\tan \phi = \tan (\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2},$$

which was to be proved.

33. Theorems relating to parallel lines.

I. *If two straight lines are parallel their slopes are equal.*

II. *The equation of the straight line parallel to a given line $y = mx + b$, and passing through a given point (x_1, y_1) is*

$$y - y_1 = m(x - x_1). \tag{13}$$

Compare with (5), Art. 23.

III. *The lines $Ax + By + C = 0$, and $A'x + B'y + C' = 0$ are parallel if $A/B = A'/B'$.*

The slopes of these two lines are respectively $-(A/B)$ and $-(A'/B')$, see Art. 30. The lines being parallel these are equal.

It follows that two equations which differ only in their absolute terms represent parallel lines, for example

$$Ax + By + C = 0, \qquad \text{and} \qquad Ax + By + D = 0.$$

IV. *The equation of the straight line parallel to $Ax + By + C = 0$, and passing through (x_1, y_1) may also be written in the form*

$$A(x - x_1) + B(y - y_1) = 0. \tag{14}$$

Equation (14) represents a line parallel to $Ax + By + C = 0$ because the coefficients of x and y in the two equations are the same, and it passes through the point (x_1, y_1) because the equation is satisfied when these values are substituted for x and y.

34. Theorems relating to perpendicular lines.

I. *If two straight lines are perpendicular to each other, the slope of one is the negative reciprocal of the slope of the other.*

If two lines are perpendicular the angle ϕ in equation (12), p. 51, is 90°, hence $\tan\phi = \infty$, and therefore the denominator of the fraction $\dfrac{m_1 - m_2}{1 + m_1m_2}$ is zero. That is $1 + m_1m_2 = 0$, and hence

$$m_2 = -\frac{1}{m_1}, \qquad \text{and} \qquad m_1 = -\frac{1}{m_2}. \tag{15}$$

From this it follows that:

II. *The equation of the straight line perpendicular to $y = mx + b$, and passing through (x_1, y_1) is*

$$y - y_1 = -\frac{1}{m}(x - x_1). \tag{16}$$

III. *The two lines $Ax + By + C = 0$ and $A'x + B'y + C' = 0$ are perpendicular if $AA' + BB' = 0$.*

The slopes of these two lines are $-(A/B)$ and $-(A'/B')$, respectively, see Art. 30, hence if they are perpendicular we have by (15)
$$-(A/B) = (B'/A'),$$
or
$$AA' + BB' = 0. \tag{17}$$

It follows from (17) that two equations $Ax + By + C = 0$, $Bx - Ay + D = 0$, in which the coefficients of x and y are interchanged and the sign of one of them is changed, always represent perpendicular lines.

IV. *The equation of the line perpendicular to $Ax + By + C = 0$ and passing through (x_1, y_1) is*

$$B(x - x_1) - A(y - y_1) = 0. \tag{18}$$

The formal proof is left to the student.

Exercises. 1. Determine the tangent of the angle between each of the following pairs of lines.

(a) $3x - y + 16 = 0, \quad x + 2y - 7 = 0.$ Ans. ± 7.

(b) $y - x = 0, \quad 7x + 12y + 2 = 0.$ Ans. $\pm \frac{19}{5}$.

(c) $x + 3y + 6 = 0, \quad 5x + 2y + 1 = 0.$

2. (a) Find the equation of the line which passes through the point (1, 2) and is parallel to $3x - y - 7 = 0$. Ans. $3x - y - 1 = 0$.

(b) Also through (0, 0) parallel to $7x + 3y + 8 = 0$.

Ans. $7x + 3y = 0$.

(c) Also through $(7, -1)$ parallel to $3x - y - 6 = 0$.

3. With the data of Ex. 2, find the equation of the straight line which passes through each given point perpendicular to the given line.

Ans. (a) $x + 3y - 7 = 0$, (b) $3x - 7y = 0$, (c) $x + 3y - 4 = 0$.

35. Perpendicular distance to a given point from a given line.

Theorem I. *The perpendicular distance to the point (x', y') from the line $x \cos \alpha + y \sin \alpha - p = 0$ is numerically equal to the expression $x' \cos \alpha + y' \sin \alpha - p$.*

Let $P(x', y')$ be the given point, and AB the given line, $x \cos \alpha + y \sin \alpha - p = 0$. Then

$$OL = p, \qquad \text{and} \qquad XOL = \alpha. \tag{i}$$

Draw MN through O parallel to AB. There are two cases.

I. When the point P and the line AB are both on the same side of MN, as in Fig. 31. Through P draw CD parallel to AB.

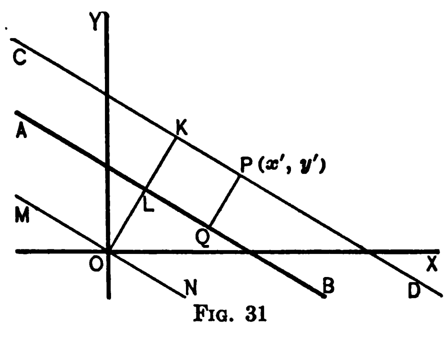


Fig. 31

The parameters of the equation of CD in the normal form are $\alpha = XOK$, the same as for AB, and $OK = p'$. Hence the equation of CD in the normal form is

$$x \cos \alpha + y \sin \alpha - p' = 0. \qquad (ii)$$

Since CD passes through the point P the coordinates of P will satisfy equation (ii), hence

$$x' \cos \alpha + y' \sin \alpha - p' = 0,$$

or

$$p' = x' \cos \alpha + y' \sin \alpha, \qquad (iii)$$

hence p' is known, since x', y', and α are known.

The length of the required perpendicular is QP, and from Fig. 31

$$QP = LK = LO + OK = OK - OL,$$

hence from (i) and (iii)

$$QP = p' - p = x' \cos \alpha + y' \sin \alpha - p. \qquad (iv)$$

II. When the point P and the line AB are on opposite sides of MN, as in Fig. 32. As before draw CD through P parallel to AB, and LOK through O perpendicular to CD. Then

$$QP = LO + OK. \qquad (v)$$

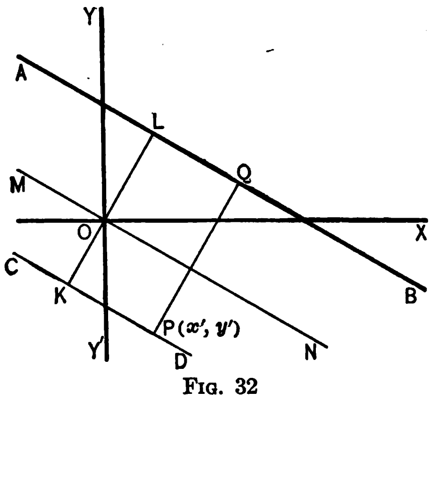


FIG. 32

Denote the length of OK by p'. The figure shows that the length of QP is numerically equal to the sum of the lengths LO and OK, and therefore since from (*i*) $LO = -p$, OK must also be taken with the negative sign, hence from (*v*)

$$QP = -p - p'. \qquad (vi)$$

Since angle $XOK = 180° + \alpha$, where $\alpha = XOL$, the parameters of CD are $180° + \alpha$ and p', and the equation of CD is

$$x \cos (180° + \alpha) + y \sin (180° + \alpha) - p' = 0,$$

or

$$-x \cos \alpha - y \sin \alpha - p' = 0. \qquad (vii)$$

The point P is on CD, therefore its coordinates (x', y') will satisfy (*vii*), and hence

$$-x' \cos \alpha - y' \sin \alpha - p' = 0,$$

or

$$p' = -(x' \cos \alpha + y' \sin \alpha). \qquad (viii)$$

Finally from (*vi*) and (*viii*)

$$QP = x' \cos \alpha + y' \sin \alpha - p,$$

In every case therefore the numerical value of the length d of the perpendicular to a point (x', y') from the line

$$x \cos \alpha + y \sin \alpha - p = 0$$

is

$$\mathbf{d = x' \cos \alpha + y' \sin \alpha - p.} \qquad (19)$$

SIGN OF THE PERPENDICULAR. From (*iv*) under Case I it is seen that QP is $+$ or $-$ according as $p' > p$, or $p' < p$. Therefore QP is positive when P is on the opposite side of AB from the origin, negative when P is in the part of the plane between AB and MN, in other words when the origin and P are on the same side of AB. When MN is between P and AB, QP is always negative, as appears from (*vi*). In this case the origin and P are always on the same side of AB. In other words:

THEOREM II. *The perpendicular d from the line AB to the point P is positive when P and the origin are on opposite sides of AB, and negative when P and the origin are on the same side of AB.*

THEOREM III. *The perpendicular distance to the point (x', y') from the line $Ax + By + C = 0$ is*

$$\frac{Ax' + By' + C}{\pm \sqrt{A^2 + B^2}},$$

where the sign of the radical is determined as in Art. 31.

The equation of a line in the form $Ax + By + C = 0$ is reduced to the normal form by dividing by $\pm \sqrt{A^2 + B^2}$, where the radical is to be given the sign opposite to that of C, as shown in Art. 31. When this division has been performed, we have from (19)

$$d = \frac{Ax' + By' + C}{\pm \sqrt{A^2 + B^2}}. \tag{20}$$

EXAMPLE. What is the perpendicular distance from the line $3x - 4y + 10 = 0$ to the point $(2, -1)$?

By formula (20) the result is

$$d = \frac{3(2) - 4(-1) + 10}{-5} = \frac{6 + 4 + 10}{-5} = -4.$$

Note that we divide by $-\sqrt{A^2 + B^2} = -\sqrt{9 + 16} = -\sqrt{25}$, because $C = +10$; also the result being -4, the distance from the given line to the given point is 4 units, and the point is on the same side of the line as the origin.

THEOREM IV. *$Ax + By + C = 0$ being any straight line, (x_1, y_1), (x_2, y_2), any two points not on the line, will lie on the same side or on opposite sides of the line according as the two numbers $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have like or unlike signs.*

The proof is left to the student.

EXAMPLES. 1. Are the points $(6, 2)$, $(-5, 1)$ on the same or opposite sides of the line $3x - 2y + 7 = 0$?

Substituting the coordinates of the given points in the left-hand side of the given equation the results are $3(6) - 2(2) + 7 = 21$, $3(-5) - 2(1) + 7 = -10$, hence the points are on opposite sides of the line. (Theorem IV.)

In the exercises which follow draw the figure in every case.

2. Find the length of the perpendicular from

(a) $3x - 4y - 7 = 0$ to $(2, 3)$, (c) $x - 5y - 6 = 0$ to $(-3, 0)$,
(b) $12x + 5y + 8 = 0$ to $(1, -4)$, (d) $2x - 3y + 7 = 0$ to $(0, 4)$.

Ans. (a) $-\frac{13}{5}$, (b) 0, (d) $+\frac{5}{13}\sqrt{13}$.

3. In each of the following cases determine whether the given points are on the same or opposite sides of the given line.

(a) $(2, 3)$, $(8, -6)$, $3x - 4y - 7 = 0$.
(b) $(1, -2)$, $(-2, 1)$, $7x - 5y + 6 = 0$.
(c) $(5, 1)$, $(-2, 3)$, $x + 2y - 9 = 0$.

4. Find the lengths of the three altitudes of the triangle whose vertices are $(6, 2)$, $(3, -5)$, $(-1, 7)$. Ans. $\frac{32}{29}\sqrt{58}$, $\frac{32}{37}\sqrt{74}$, $\frac{8}{5}\sqrt{10}$.

36. Lines through the intersection of two given lines.

Theorem. *The equation*

$$Ax + By + C + k(A'x + B'y + C') = 0, \qquad (21)$$

where k is any constant, is the equation of a straight line passing through the intersection of the two lines $Ax + By + C = 0$ and $A'x + B'y + C' = 0$.

Equation (21) is the equation of a straight line because it is of the first degree in the variables x and y.

Let (x', y') be the point of intersection of the two given lines, then their equations will both be satisfied for $x = x'$ and $y = y'$. Hence $Ax' + By' + C \equiv 0$ and $A'x' + B'y' + C' \equiv 0$. Therefore equation (21) is satisfied for $x = x'$ and $y = y'$, and hence is the equation of a line which passes through (x', y').

The number k in equation (21) is called the undetermined parameter of the line. By assigning the proper value to k, the line can be made to satisfy one other condition besides passing through the intersection of $Ax + By + C = 0$ and $A'x + B'y + C' = 0$. The exercises worked below illustrate this.

EXERCISES. 1. Determine the equation of the line which joins the origin to the point of intersection of the two lines $x - 5y - 6 = 0$ and $7x + 2y + 3 = 0$.

From (21) the equation of any line through the intersection of the two given lines will be

$$x - 5y - 6 + k(7x + 2y + 3) = 0. \qquad (i)$$

Since this line is to pass through the origin (0, 0) the values $x = 0$, $y = 0$ must satisfy its equation. Substituting in (i) $-6 + 3k = 0$, or $k = 2$. This value of k in (i) gives $15x - y = 0$ for the required equation.

2. Find the equation of the line passing through the intersection of $2x - 5y - 8 = 0$ and $3x + 2y - 9 = 0$, and perpendicular to $5x - 3y + 12 = 0$.

By (21) the required equation has the form

$$2x - 5y - 8 + k(3x + 2y - 9) = 0,$$

or

$$(3k + 2)x + (2k - 5)y - 9k - 8 = 0. \qquad (ii)$$

The slope of the line whose equation is (ii) is $-[(3k + 2)/(2k - 5)]$, and the slope of $5x - 3y + 12 = 0$ is $\frac{5}{3}$. Hence by (15), p. 52, if the two lines are to be perpendicular

$$-\frac{3k + 2}{2k - 5} = -\frac{3}{5}, \qquad \text{from which } k = -\tfrac{25}{9}.$$

Substituting this value of k in (ii) the result is

$$57x + 95y - 153 = 0.$$

3. Determine the equations of the lines passing through the intersection of $7x + 2y - 9 = 0$, $x - 3y + 2 = 0$, and also

(a) passing through the point (2, 3), Ans. $2x - y - 1 = 0$.
(b) having the slope $\frac{2}{3}$, Ans. $2x - 3y + 1 = 0$.
(c) parallel to $2x + 5y + 6 = 0$,
(d) perpendicular to $2x + 5y + 6 = 0$.

37. Imaginary points and lines.

I. DEFINITION. *Two imaginary points whose coordinates differ only in the signs of the imaginary parts are called* ***conjugate imaginary points.***

For example, $(2 + 5i, 3 + 8i)$ and $(2 - 5i, 3 - 8i)$, are a pair of conjugate imaginary points.

II. THEOREM. *If an imaginary point lies on a real line $Ax + By + C = 0$, then the conjugate imaginary point lies on the same line.*

If $(a + bi, c + di)$ is a point on $Ax + By + C = 0$, then

$$A(a + bi) + B(c + di) + C = 0,$$

or

$$Aa + Bc + C + i(Ab + Bd) = 0, \qquad (i)$$

hence

$$Aa + Bc + C = 0, \qquad \text{and} \qquad Ab + Bd = 0. \qquad (ii)$$

See IV (*c*), p. viii.

Then from (*ii*) it must also follow that

$$Aa + Bc + C - i(Ab + Bd) = 0,$$

or

$$A(a - bi) + B(c - di) + C = 0.$$

Hence the point $(a - bi, c - di)$, conjugate to $(a + bi, c + di)$, also satisfies the equation $Ax + By + C = 0$.

III. THEOREM. *On every imaginary line there is one and only one real point.*

Let

$$(p_1 + q_1 i)x + (p_2 + q_2 i)y + p_3 + q_3 i = 0$$

be the equation of an imaginary line. This equation may be written in the form

$$p_1 x + p_2 y + p_3 + i(q_1 x + q_2 y + q_3) = 0. \qquad (iii)$$

In general there will be one and only one pair of values of x and y which satisfy simultaneously the two equations

$$p_1 x + p_2 y + p_3 = 0, \qquad q_1 x + q_2 y + q_3 = 0, \qquad (iv)$$

and these values are always real. They will also satisfy (*iii*), hence they are the coordinates of the only real point on the

imaginary line (*iii*). This point is the intersection of the two real lines whose equations are given in (*iv*).

SPECIAL CASES. If in (*iii*)

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = k, \qquad \text{and} \qquad \frac{p_3}{q_3} \neq k$$

the two corresponding equations (*iv*) will not be satisfied by finite values of x and y. In that case the real point on the imaginary line (*iii*) is said to be at infinity, and the real lines (*iv*) are parallel.

If

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \frac{p_3}{q_3} = k,$$

then $p_1 = kq_1$, $p_2 = kq_2$, $p_3 = kq_3$. These values substituted in (*iii*) give

$$k(q_1x + q_2y + q_3) + i(q_1x + q_2y + q_3) = 0,$$

from which, dividing by $k + i$, we have

$$q_1x + q_2y + q_3 = 0,$$

so that in this case the line (*iii*) is not an imaginary line.

IV. DEFINITION. *Imaginary lines whose equations differ only in the signs of their imaginary terms are called conjugate imaginary lines.*

For example

$$(p_1 - q_1i)x + (p_2 - q_2i)y + p_3 - q_3i = 0 \qquad (v)$$

is the conjugate of the line whose equation is (*iii*).

The coordinates of the real point which satisfy (*iii*) will also obviously satisfy (*v*), hence:

V. THEOREM. *A pair of conjugate imaginary lines meet in a real point.*

By III above, this is the only real point on either of them.

38. Higher equations which represent straight lines.

It was shown in Art. 18 that if the equation $f(x, y) = 0$ can be expressed in the form

$$f_1(x, y) \cdot f_2(x, y) \cdot f_3(x, y) \cdots = 0, \tag{i}$$

the locus corresponding to $f(x, y) = 0$ consists of all the separate loci corresponding to

$$f_1(x, y) = 0, \quad f_2(x, y) = 0, \quad f_3(x, y) = 0, \quad \text{etc.} \tag{ii}$$

Hence if $f(x, y) = 0$ is of the nth degree in x and y, and the equations (*ii*) are all of the first degree, the equation $f(x, y) = 0$ will represent n straight lines. These n lines will not necessarily be all distinct nor all real for some of the factors in (*ii*) may be equal to each other, and some may be imaginary.

For example the equation

$$(x - 3y + 2)^2(x^2 - y^2)(x^2 + y^2) = 0$$

corresponds to two coincident real lines $x - 3y + 2 = 0$, two other real lines $x + y = 0$, $x - y = 0$, and the conjugate imaginary lines $x + iy = 0$, $x - iy = 0$.

Exercises on Chapter III

Normal Exercises

1. Find the equation of the straight line determined by each of the following sets of quantities

(*a*) Two points $(-2, 4)$, $(3, -7)$, (*d*) $a = -3$, $b = 5$,
(*b*) $m = -\frac{1}{2}$, $b = -3$, (*e*) $p = \frac{3}{2}$, $\alpha = 30°$.
(*c*) $m = -2$, one point $(-2, 5)$,

2. Find the equation of the straight line determined by each of the following sets of quantities, the angle between the axes being 60°

(*a*) Two points $(-2, 4)$, $(3, -7)$, (*c*) $\theta = 30°$, $b = 4$,
(*b*) $\theta = 30°$, one point $(-2, 5)$, (*d*) $a = -3$, $b = 5$.

Ans. (*b*) $x - y + 7 = 0$.

3. Find the parameters a, b, p, m, and α of the following lines

(*a*) $2x + 4y + 7 = 0$, (*c*) $x + y = 0$,
(*b*) $5x - 6y = 10$, (*d*) $y = 2$.

4. Find the angle which line (*a*) of Exercise 3 makes with each of lines (*b*), (*c*), and (*d*).

5. Find the equation of the two lines which pass through the point of intersection of the lines $3x - 4y - 18 = 0$ and $2x + y - 1 = 0$ and which are respectively parallel and perpendicular to the line $x + 3y - 9 = 0$, (*a*) by using the coordinates of the point of intersection, (*b*) without finding the point of intersection.

6. Find the distance of each of the points $(-2, 0)$, $(0, 4)$, $(2, 1)$, $(-2, -4)$, $(2, 4)$, from the line $3x - 4y - 8 = 0$. Which of these points are above the given line? Which below? Which are on the same side of the line as the origin?

7. Find the coordinates of a pair of conjugate imaginary points lying on the line $4x - y + 2 = 0$.

8. Find the coordinates of the real point which lies on the imaginary line $(4 + i)x - (3 - 2i)y - (20 - 6i) = 0$. What is the equation of the conjugate imaginary line? What are the coordinates of the point of intersection of these two imaginary lines?

9. Construct the loci of the following equations, and find the equation of each separate part of each locus

(*a*) $x^2 - y^2 - 4x + 4 = 0$, (*b*) $x^2 - 4y^2 = 0$, (*c*) $x^4 - y^4 = 0$.

General Exercises

10. Given the vertices $(4, 3)$, $(2, -2)$, $(-3, 4)$ of a triangle, find the equations of the sides and of the medians, and the coordinates of the point of meeting of the latter. This point is called the centroid of the triangle. Ans. Centroid $= (1, \frac{5}{3})$.

11. Determine b so that the line $y = mx + b$ shall pass through the point (x_1, y_1), and hence derive equation (5), p. 41.

12. Given the vertices $(1, 9)$, $(11, -3)$, $(-3, -1)$ of a triangle, find the equations of the perpendiculars from each vertex to the opposite side. Find also the lengths of these perpendiculars, and the coordinates of their point of intersection. This point is called the orthocenter of the triangle. Ans. Orthocenter $= (-\frac{3}{37}, \frac{53}{37})$.

13. Given $kx + 3y + 8 = 0$, where k is an undetermined parameter. Determine k so that the line shall (*a*) pass through $(-2, 3)$, (*b*) be parallel to OX, (*c*) have slope $\frac{2}{3}$. Ans. (*a*) $k = \frac{17}{2}$.

14. Given $7x + 2y + k = 0$, where k is an undetermined parameter.

Determine k so that the line shall (a) pass through point $(3, -1)$, (b) have $a = b + 3$, (c) have $p = 2$. Ans. (b) $k = \frac{42}{5}$.

15. Given the vertices $(5, 3)$, $(-3, 1)$, $(2, -6)$ of a triangle, find the equations of the perpendicular bisectors of the three sides, and the coordinates of their point of intersection. This point is called the circumcenter of the triangle. Ans. Circumcenter $= (\frac{19}{11}, -\frac{10}{11})$.

16. Given the two lines $3x - 5y + 6 = 0$, $2x + y - 9 = 0$. Find the equation of the line which passes through the point of intersection of these lines (without finding the coordinates of this point), and which

(a) passes through the point $(-6, 1)$.
(b) is perpendicular to $5x + 2y - 7 = 0$.
(c) is parallel to $3x + y - 17 = 0$.
(d) has its x-intercept twice its y-intercept.
(e) is parallel to OY.
(f) has $p = 3$.
(g) has $b = 4$.
(h) has $\alpha = 90°$.

17. The three sides of a triangle are $5x - 3y + 22 = 0$, $6x + 5y - 8 = 0$, $x + 8y + 13 = 0$, find the equations of the perpendiculars from the vertices to the opposite sides, and the coordinates of the orthocenter (see Ex. 12), *without* finding the coordinates of the vertices of the triangle. Ans. Orthocenter $= (-\frac{101}{43}, \frac{52}{43})$.

18. Show that the equations of the bisectors of the two pairs of vertical angles formed by the lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$ are

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} \pm \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = 0,$$

and prove that these two bisectors are perpendicular to each other.

19. The three sides of a triangle are $3x + 4y - 10 = 0$, $8x - 6y + 3 = 0$, $12x + 5y + 15 = 0$. Find the equations of the bisectors of the three angles of the triangle, and the center of the inscribed circle. Ans. $(-\frac{13}{20}, \frac{31}{20})$.

20. Prove that the medians of any triangle meet in one point. [Use oblique axes, taking two sides of the triangle as axes of coordinates.]

21. Prove that the perpendicular bisectors of the sides of a triangle meet in one point. [Take the base and the perpendicular to it at one extremity as axes of coordinates.]

22. Prove that the perpendiculars let fall from the vertices of a triangle to the opposite sides meet in one point. [Take the axes as in Ex. 21.]

23. Find the locus of the point whose distance from $12x - 5y - 16 = 0$ is twice its distance from $3x + 4y - 5 = 0$.

Ans. $138x + 79y - 210 = 0$, $18x + 129y - 50 = 0$.

24. Find the locus of the point whose distance from $Ax + By + C = 0$ is k times its distance from $A'x + B'y + C' = 0$.

25. Determine how many real points lie on each of the following lines and where possible find their coordinates:

(*a*) $(4 - 2i)x + (3i - 3)y - 2 + 4i = 0$,
(*b*) $(4 - 8i)x + (6i - 3)y - 2 = 0$,
(*c*) $x + iy = 1$,
(*d*) $(3 - 9i)x - (6i - 2)y + 5 - 15i = 0$.

26. Prove that $5x^2 + 13xy - 6y^2 = 0$ is the equation of two straight lines intersecting at the origin. See III, p. viii.

27. Prove that $Ax^2 + Hxy + By^2 = 0$ in which A, B, and H are real constants represents a pair of straight lines passing through the origin, and that these two lines are real and distinct, real and coincident, or imaginary, according as $H^2 - 4AB$ is positive, zero, or negative. See III, p. viii.

CHAPTER IV

THE CIRCLE

39. Equation of the circle in rectangular coordinates.

DEFINITION. *A* **circle** *is the locus of a point which moves in a plane so as always to be at a constant distance from a fixed point of the plane. The fixed point is called the* **center** *of the circle, and the constant distance is called the* **radius**.

PROBLEM. *To find the equation of the circle with radius r, and center at the point (d, e).*

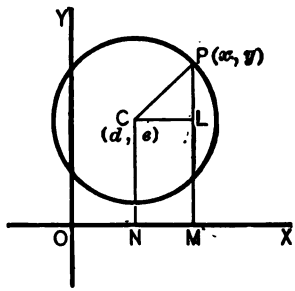


FIG. 33

Let $C(d, e)$ be the center, and $P(x, y)$ any point on the circle. Then from the definition $CP = r$. By (3), p. 6, the distance CP is $\sqrt{(x-d)^2+(y-e)^2}$. Hence

$$\sqrt{(x-d)^2+(y-e)^2} = r$$

or

$$(x-d)^2+(y-e)^2 = r^2, \tag{1}$$

which is the required equation.

By inserting the appropriate values of d, e and r, equation (1) will give the equation of any circle.

Conversely any equation in x and y which is reducible to the form (1) is the equation of a circle in rectangular coordinates.

(*a*) *The equation of the circle, the origin being at the center.* See Fig. 34. In this case $d = e = 0$, and hence equation (1) becomes

$$x^2 + y^2 = r^2. \tag{2}$$

6

(b) *The equation of the circle through the origin whose center is on the positive portion of the X-axis.* See Fig. 35.

Here we have $d = r$, and $e = 0$, hence equation (1) becomes

$$(x - r)^2 + y^2 = r^2,$$

or,

$$y^2 = 2rx - x^2. \tag{3}$$

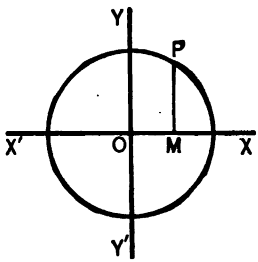


FIG. 34

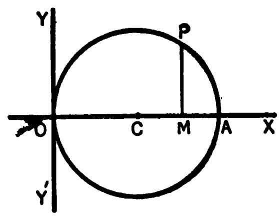


FIG. 35

40. The Equation $Ax^2 + Ay^2 + Gx + Fy + C = 0$.

I. THEOREM. *The equation of a circle can always be put in the form* $Ax^2 + Ay^2 + Gx + Fy + C = 0$, *where* $A \neq 0$.

Expanding equation (1), Art. 39, it takes the form

$$x^2 + y^2 - 2dx - 2ey + d^2 + e^2 - r^2 = 0,$$

or

$$x^2 + y^2 + Lx + My + N = 0, \tag{4}$$

where

$$L = -2d, \qquad M = -2e, \qquad N = d^2 + e^2 - r^2.$$

If equation (4) be multiplied by any constant A, and if in the result we put $AL = G$, $AM = F$, and $AN = C$, the equation takes the form

$$Ax^2 + Ay^2 + Gx + Fy + C = 0. \tag{5}$$

Note that equation (4) is just as general as equation (5), because, since $A \neq 0$, the latter can always be reduced to the former by dividing by A.

II. THEOREM. *Equation* (4) *always represents a real circle when* $N < \frac{1}{4}(L^2 + M^2)$.

Equation (4) can be put in the form

$$(x^2 + Lx + \tfrac{1}{4}L^2) + (y^2 + My + \tfrac{1}{4}M^2) = \tfrac{1}{4}(L^2 + M^2) - N,$$

or

$$(x + \tfrac{1}{2}L)^2 + (y + \tfrac{1}{2}M)^2 = \tfrac{1}{4}(L^2 + M^2) - N. \qquad (i)$$

This equation is in the form (1), Art. 39, if $\frac{1}{4}(L^2 + M^2) - N > 0$, or $N < \frac{1}{4}(L^2 + M^2)$, and it then represents a circle with center at the point whose coordinates are

and radius

$$\left.\begin{aligned} d = -\tfrac{1}{2}L, \qquad e = -\tfrac{1}{2}M \\ r = \sqrt{\tfrac{1}{4}(L^2 + M^2) - N}. \end{aligned}\right\} \qquad (6)$$

III. POINT CIRCLE, IMAGINARY CIRCLE.

(*a*) If $N = \frac{1}{4}(L^2 + M^2)$, $r = 0$, and equation (*i*) takes the form

$$(x + \tfrac{1}{2}L)^2 + (y + \tfrac{1}{2}M)^2 = 0, \quad \text{or} \quad (x - d)^2 + (y - e)^2 = 0. \quad (ii)$$

The most obvious interpretation of this equation is that of a circle of zero radius, the *point circle* $(-\frac{1}{2}L, -\frac{1}{2}M)$ or (d, e), since these are the only real coordinates which satisfy the equation.*

(*b*) If $N > \frac{1}{4}(L^2 + M^2)$, r^2 is negative, and equation (*i*) is that of an imaginary locus with no real points. See Art. 16 (*b*). The locus is called an *imaginary circle.*

The equation $x^2 + y^2 + Lx + My + N = 0$ therefore represents a real circle, a point circle, or an imaginary circle according as $N \lesseqgtr \frac{1}{4}(L^2 + M^2)$. Hence:

IV. THEOREM. *The general equation of the second degree in* x *and* y, $Ax^2 + Hxy + By^2 + Gx + Fy + C = 0$, *always repre-*

* Since the equation $(x - d)^2 + (y - e)^2 = 0$ may be written in the form $[(x - d) + i(y - e)][(x - d) - i(y - e)] = 0$ another interpretation is that the equation represents a pair of imaginary lines meeting at the real point (d, e). See Arts. 16 (*c*), 37 V, and 38.

sents a real circle, a point circle, or an imaginary circle, when $H = 0$, *and* $A = B \neq 0$.

When in practice it becomes necessary to examine an equation of the form (4) or (5) to determine the nature of the locus represented, it is best actually to reduce it to the form (1). The following example illustrates the process.

EXAMPLE. Determine whether the equation

$$3x^2 + 3y^2 - 18x + 12y - 69 = 0$$

represents a real circle, and if so determine its center and radius.

Dividing the given equation by 3

$$x^2 + y^2 - 6x + 4y - 23 = 0,$$

which is in the form (4). Adding 9 and 4 to both sides in order to complete the squares of the terms in x and y,

$$x^2 - 6x + 9 + y^2 + 4y + 4 = 36,$$

or

$$(x - 3)^2 + (y + 2)^2 = 36,$$

which is in the form (1). By comparison $d = 3$, $e = -2$, $r = 6$. In other words the given equation is that of a real circle whose center is at the point $(3, -2)$, and whose radius is 6 units.

EXERCISES. 1. Write the equation of each of the following circles, and construct the curve.

(*a*) Center at (3, 4), radius 6. (*b*) Center at $(-2, -5)$, radius 4.

(*c*) Center at $(4, -2)$, radius 7.

2. The radius of a circle is 5. Write its equation, and construct the circle for each of the following positions of the center, (3, 4), $(-3, 4)$, (0, 0), (0, 5), $(0, -5)$, (5, 5).

3. Determine the special forms assumed by equations (1), p. 65, and (4), p. 66, for circles whose positions are defined as follows.

(*a*) Passing through (0, 0), with center on the negative portion of the X-axis. Ans. $y^2 = -2rx - x^2$.

(*b*) Passing through (0, 0), with center on the Y-axis.

(*c*) Tangent to the X-axis, and center above the X-axis. Ans. $x^2 + y^2 - 2dx - 2ry + d^2 = 0$.

(*d*) Tangent to both axes, with center in the second quadrant.

4. Determine which of the following equations represent real circles. Find the center and radius of each of these and construct the curve.

(*a*) $x^2 + y^2 - 4x + 6y - 3 = 0.$ (*c*) $x^2 + y^2 + 3x + 7y + 15 = 0.$
(*b*) $x^2 + y^2 - 6x - 2y + 10 = 0.$ (*d*) $9x^2 + 9y^2 + 12x - 6y - 31 = 0.$

Ans. (*a*) $d = 2,\ e = -3,\ r = 4.$

5. Determine the equations of the following circles, and construct the curves.

(*a*) Passing through the point (3, 4), center at the origin.
(*b*) Passing through the origin, center at (3, 4).
(*c*) Passing through (2, 4), center at (− 1, 0).

Ans. (*b*) $x^2 + y^2 - 6x - 8y = 0,$ (*c*) $x^2 + y^2 + 2x - 24 = 0,$

41. The circle through three given points.—Three points not in the same straight line determine one and only one circle. The process of finding the equation of the circle will be illustrated by an example.

Find the equation of the circle passing through the three points (5, 7), (2, − 1), (− 3, 3). Either equation (1) or equation (4) may be used. The latter is usually the more convenient and will be used here. The process consists in finding the values of L, M, N for which equation (4) will be satisfied when x and y have each of the three given pairs of values.

Hence substituting in turn these values for x and y in (4), three simultaneous equations in L, M, and N are obtained:

$$\begin{aligned} 74 + 5L + 7M + N &= 0, \\ 5 + 2L - M + N &= 0, \\ 18 - 3L + 3M + N &= 0. \end{aligned}$$

Solving these for L, M, and N, the results are

$$L = -\tfrac{43}{13}, \qquad M = -\tfrac{96}{13}, \qquad N = -\tfrac{75}{13},$$

and substituting these in (4)

$$x^2 + y^2 - \tfrac{43}{13}x - \tfrac{96}{13}y - \tfrac{75}{13} = 0,$$

or

$$13x^2 + 13y^2 - 43x - 96y - 75 = 0,$$

which is the equation of the required circle. Let the student verify the result by substituting the coordinates of the given points in this equation.

The general solution of this problem is most conveniently performed with the aid of determinants. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be three given points not in the same straight line. Substituting these in turn for x and y in the equation

$$x^2 + y^2 + Lx + My + N = 0,$$

we have

$$x_1^2 + y_1^2 + Lx_1 + My_1 + N = 0,$$

$$x_2^2 + y_2^2 + Lx_2 + My_2 + N = 0,$$

$$x_3^2 + y_3^2 + Lx_3 + My_3 + N = 0.$$

If the last three of these equations be solved for L, M, N, and the results so obtained be substituted in the first equation, the result will be the equation of the required circle. This is the process which was used in the example above. By determinants the same result is obtained by expressing the fact that all *four* equations are satisfied simultaneously by the same values of the *three* quantities L, M, N, which gives

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{7}$$

42. Distance from a point to a circle.

THEOREM I. *If $P(x_1, y_1)$ is any point in the plane of the circle $(x - d)^2 + (y - e)^2 = r^2$, outside of the circle, the distance from P to the circle, measured on one of the two tangents through P, is*

$$\sqrt{(x_1 - d)^2 + (y_1 - e)^2 - r^2}. \tag{8}$$

In Fig. 36 let $C = (d, e)$, and $P = (x_1, y_1)$, then

$$\overline{CP}^2 = (x_1 - d)^2 + (y_1 - e)^2.$$

The triangle PCN is right-angled at N, and $CN = r$, hence

$$PN = \sqrt{\overline{CP}^2 - \overline{CN}^2} = \sqrt{(x_1 - d)^2 + (y_1 - e)^2 - r^2}.$$

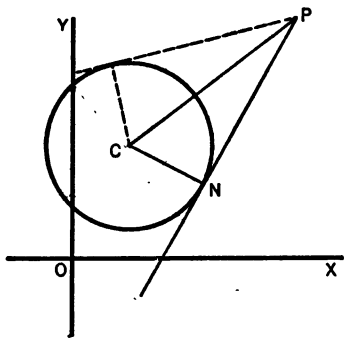


FIG. 36

Similarly if the equation of the circle is

$$x^2 + y^2 + Lx + My + N = 0$$

the distance from P to the circle is

$$\sqrt{x_1^2 + y_1^2 + Lx_1 + My_1 + N}. \qquad (9)$$

Let the student prove this and also the following theorem.

THEOREM II. *The point* (x_1, y_1) *is outside, on, or inside the circle* $x^2 + y^2 + Lx + My + N = 0$ *according as the expression* $x_1^2 + y_1^2 + Lx_1 + My_1 + N$ *is positive, zero, or negative.*

EXERCISES. 1. Find the equations of the circles determined by the following sets of points. Construct each circle from its equation, and check the result by noting whether the circle passes through the given points.

(*a*) (5, 7), (2, 8), (7, 3). (*c*) (1, 5), (9, − 7), (− 4, 6).
(*b*) (0, 0), (3, 1), (− 2, − 4). (*d*) (0, 0), (0, 6), (3, − 3).

Ans. (*a*) $x^2 + y^2 - 4x - 6y - 12 = 0$.

2. Given the circle $x^2 + y^2 + 4x - 6y + 4 = 0$, and the points (0, 0), (− 2, 1), (1, − 3), (− 1, 3), (1, 3), (− 4, 2). Determine which of the points are inside and which are outside the circle. Check the results graphically.

3. For each point of Exercise 2 which is outside the given circle, find the length of the tangents from the point to the circle.

43. Equation of a circle through the points of intersection of two circles.—Let

$$x^2 + y^2 + L_1x + M_1y + N_1 = 0, \qquad (i)$$

$$x^2 + y^2 + L_2x + M_2y + N_2 = 0 \qquad (ii)$$

be the equations of two circles. Form a third equation by adding one of these to k times the other, where k is any constant. The result is

$$(x^2 + y^2 + L_1x + M_1y + N_1) + k(x^2 + y^2 + L_2x + M_2y + N_2) = 0,$$

or

$$(1 + k)x^2 + (1 + k)y^2 + (L_1 + kL_2)x + (M_1 + kM_2)y + (N_1 + kN_2) = 0. \qquad (10)$$

For all values of k, except $k = -1$, this equation is in the form (5), p. 66, and hence represents a circle. If $k = -1$, (10) is an equation of the first degree, and therefore represents a straight line.

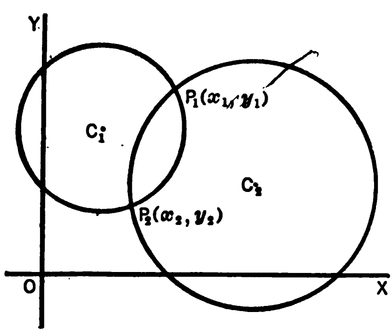


FIG. 37

THEOREM. *For all values of k, equation* (10) *is the equation of a locus (circle or straight line) which passes through the intersections of the circles given by equations* (i) *and* (ii).

Let C_1, C_2 be the centers of the circles represented by equations (i) and (ii), and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ their points of intersection. Considering the point P_1, we have from (i) and (ii)

$$x_1^2 + y_1^2 + L_1x_1 + M_1y_1 + N_1 \equiv 0,$$

$$x_1^2 + y_1^2 + L_2x_1 + M_2y_1 + N_2 \equiv 0,$$

and hence the coordinates of P_1 will satisfy equation (10). In other words the locus represented by this equation passes through P_1. Similarly it passes through P_2.

If the circles represented by equations (*i*) and (*ii*) meet in imaginary points (that is if they do not visibly intersect), equation (10) will still represent a locus passing through their imaginary points of intersection, for the argument above holds good whether P_1 and P_2 have real or imaginary coordinates.

EXERCISES ON CHAPTER IV

Normal Exercises

1. Write the equations of the circles with radius 4, whose centers are respectively $(-3, -2)$, $(0, -5)$, $(4, 0)$, $(0, 0)$, and construct each circle from its equation.

2. Determine which of the following equations represent real circles, and find the center and radius of each of these.

(*a*) $x^2 + y^2 - 2x + 4y - 4 = 0$, (*c*) $x^2 + y^2 + 2x + 4y + 10 = 0$,
(*b*) $4x^2 + 4y^2 = 25$, (*d*) $36x^2 + 36y^2 - 48x + 108y + 97 = 0$.

3. Find the equations of the circles passing through the following sets of points.

(*a*) $(2, 2)$, $(3, 1)$, $(-5, -5)$, (*b*) $(2, 1)$, $(3, -2)$, $(-4, 0)$.

Ans. (*a*) $x^2 + y^2 + 2x + 4y - 20 = 0$.

4. Find the length of the tangents from each of the points $(0, 1)$, $(2, 3)$, $(4, 2)$, $(-3, -1)$ to the circle $x^2 + y^2 - 4x + 8y - 5 = 0$.

Ans. 2, $2\sqrt{6}$, etc.

5. Determine which of the points $(0, 0)$, $(1, -1)$, $(-2, 3)$, $(-1, 3)$, $(0, \frac{5}{2})$ are inside the circle $4x^2 + 4y^2 + 12x - 20y + 25 = 0$.

6. Write the equation which represents all circles passing through the points of intersection of the circles $x^2 + y^2 - 2x - 2y - 23 = 0$, and $x^2 + y^2 - 16x - 9y + 68 = 0$.

7. Determine the equation of the circle which passes through the intersections of the two circles of Exercise 6, and also through the origin.

Ans. $13x^2 + 13y^2 - 72x - 49y = 0$.

General Exercises

8. Find the equations of the circles determined by the following sets of conditions.

(*a*) Center at the intersection of $x + 3y + 7 = 0$ and $2y - 3x + 12 = 0$ and passing through $(-1, 1)$.

Ans. $x^2 + y^2 - 4x + 6y - 12 = 0$.

(*b*) Center at $(1, 3)$, and tangent to $2x + y + 5 = 0$.

(*c*) Center on $x + y - 5 = 0$, and tangent to both coordinate axes.

Ans. $4x^2 + 4y^2 - 20x - 20y + 25 = 0$.

(*d*) Circumscribed about the triangle whose sides have the equations

$$x + y - 4 = 0,\ 4x - y - 6 = 0,\ 2x - y - 8 = 0.$$

(*e*) Inscribed in the triangle whose vertices are $(0, 0)$, $(0, 6)$, $(8, 0)$.

Ans. $x^2 + y^2 - 4x - 4y + 4 = 0$.

(*f*) Having the line from $(2, -3)$ to $(-4, 7)$ as a diameter.

9. What system of circles is represented by each of the following equations.

(*a*) $x^2 + y^2 + Lx + 2y = 0$, (*c*) $4x^2 + 4y^2 + 4Lx + L^2 - 16 = 0$,
(*b*) $x^2 + y^2 + Lx + Ly = 0$, (*d*) $x^2 + y^2 - 2rx - 2ry + r^2 = 0$.

Ans. (*a*) Circles passing through the origin, and having their centers on $y + 1 = 0$.

10. Of all circles which pass through the intersections of $x^2 + y^2 + 6x + 8y - 144 = 0$ and $x^2 + y^2 - 10x - 8y + 16 = 0$, find the equation of the one which (*a*) passes through the origin, (*b*) has its center on the X-axis, (*c*) has its center on the line $x - 2y - 2 = 0$.

Ans. (*a*) $5x^2 + 5y^2 - 42x - 32y = 0$, (*b*) $x^2 + y^2 - 2x - 64 = 0$.

11. Prove by the methods of analytic geometry that an angle inscribed in a semi-circle is a right angle.

12. Show by means of their equations that a straight line can not cut a circle in more than two points.

13. Find the equation of the locus of a point whose distance from the origin is always twice its distance from $(2, 3)$. What curve is it?

Ans. $3x^2 + 3y^2 - 16x - 24y + 52 = 0$.

14. Find the locus of a point whose distance from $3x + 4y - 1 = 0$ is always equal to the square of its distance from $(2, 3)$.

Ans. $5x^2 + 5y^2 - 23x - 34y + 66 = 0$.

15. Find the locus of a point the square of whose distance from $x + y - 2 = 0$ is equal to the area of the rectangle formed by drawing perpendiculars from the point to the axes.

Ans. $x^2 + y^2 - 4x - 4y + 4 = 0$.

CHAPTER V

THE CONIC SECTIONS

44. Historical note.—The Greek geometers, after studying the geometry of the right line, plane, circle, sphere, cylinder and cone, turned their attention to three curves called the ellipse, parabola, and hyperbola, which they derived by cutting right circular cones by planes. For this reason these curves are called *conic sections*, or briefly, *conics*. In applying to them the methods of analytic geometry, however, we prefer to disregard their connection with the cone and define them as plane curves, using for this purpose certain characteristic properties from which their equations can be easily derived.

45. Definitions.

I. CONIC, FOCUS, DIRECTRIX. *A* **conic** *is the locus of a point which moves so that the ratio of its distance from a fixed point, called the* **focus**, *to its distance from a fixed line, called the* **directrix**, *remains constant, the point being always in the plane of the focus and directrix.*

II. ECCENTRICITY. *The constant ratio in* I *is called the* ***eccentricity*** *of the conic. It is represented by e.*

III. ELLIPSE. *If $e < 1$ the conic is called an* **ellipse.**

IV. PARABOLA. *If $e = 1$ the conic is called a* **parabola.**

V. HYPERBOLA. *If $e > 1$ the conic is called a* **hyperbola.**

46. Problem.—*To construct a conic having given focus, directrix, and eccentricity.*

I. THE ELLIPSE. Let F, Fig. 38, be the given focus and DD' the given directrix of a conic with eccentricity $e < 1$. Through F draw EG perpendicular to DD'. Construct the angle GEH, so that $\tan GEH = e$, and draw $GEH' = GEH$. Since $e < 1$, the

angle $GEH < 45°$. Draw a series of parallels perpendicular to EG. On any one of them, as KK', mark with dividers the

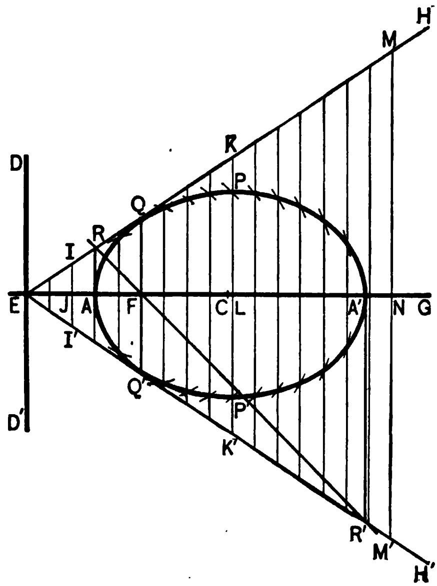


FIG. 38

points P, P', so that $FP = FP' = LK$. Then P and P' are points on the ellipse, because, EL being the distance of P from the directrix,

$$\frac{FP}{EL} = \frac{LK}{EL} = \tan GEH = e.$$

The same is true of P'. As many points of the ellipse as may be desired can be constructed in this way.

As this construction is continued with successive parallels farther and farther from F a point will be reached beyond which the construction ceases to be possible. For example on $M'M$ there is no point whose distance from F equals NM, and

hence $M'M$ does not intersect the ellipse. The same is true of a line such as $I'I$ sufficiently far from F on the other side. In other words the ellipse is limited in extent in both directions from F. It is obvious also that the ellipse lies wholly between the lines EH, EH', and that it touches these lines at Q and Q', where the perpendicular to EG through F cuts them.

To find the points where the ellipse cuts EG draw RR' through F, making angle $EFR = 45°$, cutting EH, EH' in R, R' respectively. From R and R' let fall perpendiculars to EG, meeting it at A and A'. These will be the required points, because

$$\frac{AF}{EA} = \frac{AR}{EA} = \tan GEH = e,$$

and

$$\frac{FA'}{EA'} = \frac{R'A'}{EA'} = \tan GEH' = e.$$

From this discussion it is clear that the ellipse is a closed curve symmetrical with respect to EG, and lying in that part of the plane between A and A'.

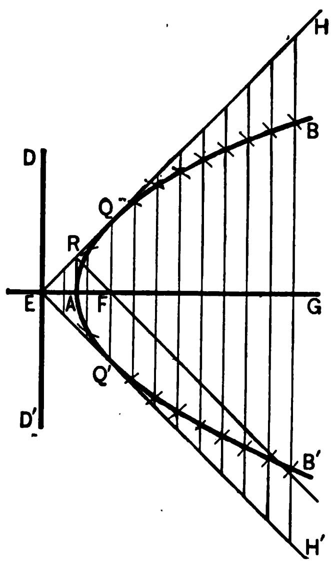


Fig. 39

Exercises. 1. Construct the ellipse for which $e = \frac{1}{3}$, and $EF = 2$ inches.

2. Construct the ellipse for which $e = \frac{1}{2}$, and $EF = 1\frac{1}{2}$ inches.

3. Construct the ellipse for which $e = \frac{3}{4}$, and $EF = 1$ inch.

II. The Parabola. The method of construction is the same as for the ellipse. Let F be the focus and DD' the directrix. Since $e = 1$ the angles GEH and GEH' are each 45°, and the

line RF is parallel to EH'. The point A is on the curve, but there is no corresponding point A'.

In making the construction the following facts with reference to the parabola become apparent.

(*a*) The point A bisects the distance between the focus and directrix.

(*b*) The distances FQ, FQ' are equal in length to EF, and hence to $2AF$.

(*c*) The parabola is an open curve symmetrical with respect to EG, and extending indefinitely from A, on through B and B'.

EXERCISE. Construct the parabola for which $EF = 1$ inch.

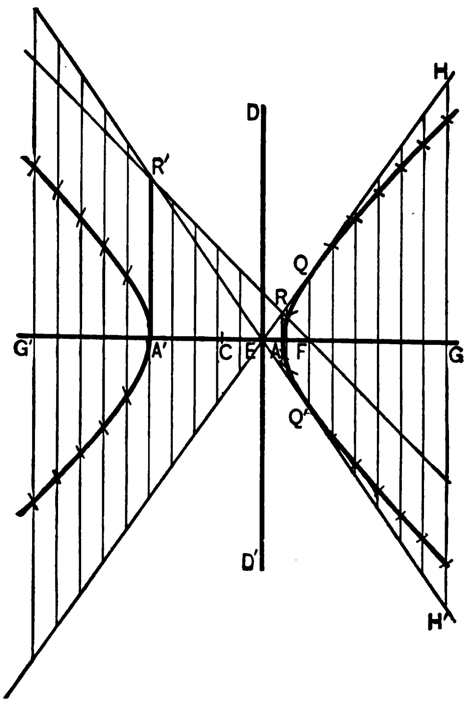


FIG. 40

III. The Hyperbola. Let F be the focus and DD' the directrix. The construction is the same as for the ellipse and parabola. Since $e > 1$, $GEH = GEH' > 45°$, and the line RF cuts EH and EH' on opposite sides of the directrix. Hence A and A' are also on opposite sides of the directrix. The curve consists of two parts extending indefinitely to right and left from A and A', and no part of it lies between the two lines drawn through A and A' parallel to the directrix. Like the ellipse and parabola it is symmetrical with respect to EG.

Exercises. 1. Construct the hyperbola for which $e = \frac{4}{3}$, and $EF = \frac{3}{4}$ inch.

2. Construct the hyperbola for which $e = 2$, and $EF = \frac{1}{2}$ inch.

47. Definitions.

I. *The perpendicular to the directrix through the focus of a conic, the line EG in Figs.* 38, 39, 40, *is called the* **principal axis.**

II. *The points A, A′ in the ellipse and hyperbola, and the point A in the parabola, where these curves respectively cut the principal axis, are called the* **vertices.**

III. *In the ellipse and hyperbola the point C which bisects AA′ is called the* **center** *of the curve.*

IV. *The chord through the focus perpendicular to the principal axis, Q′Q in Figs.* 38, 39, 40, *is called the* **latus rectum.** Note especially Art. 46, II (b).

48. The equation of any conic.—By suitably choosing the coordinate axes the equation of any conic can be derived by translating the definition Art. 45, I, into the language of algebra. The principal axis is taken as the X-axis, and the directrix as the Y-axis. The constants

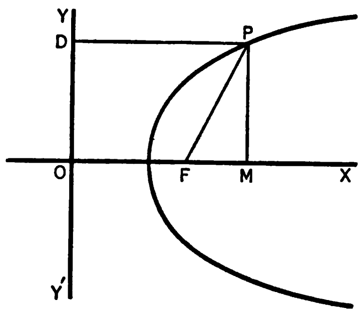


Fig. 41

to be used are the given distance $OF = p$, and the eccentricity e.

Let P be any representative point on the conic, then from the definition Art. 45, I,

$$\frac{FP}{DP} = \frac{FP}{OM} = e, \tag{i}$$

but $FP = \sqrt{(x - p)^2 + y^2}$, from (3), p. 6, since $P = (x, y)$, and $F = (p, 0)$; and $OM = x$. Hence substituting in (i)

$$\frac{\sqrt{(x - p)^2 + y^2}}{x} = e,$$

or

$$(x - p)^2 + y^2 = e^2x^2. \tag{1}$$

49. Transformation of coordinates.—It is sometimes necessary or desirable to change the axes of coordinates to which a locus is referred. This change of axes has no effect upon the locus itself, but it changes the equation.

PROBLEM. *To derive the formulas for transforming the equation of a locus from one pair of axes to another pair, when the new X-and Y-axes are parallel respectively to the original X- and Y-axes.*

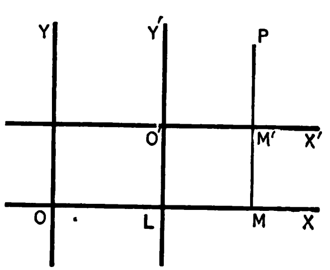


FIG. 42

Let OX, OY be the original axes of coordinates, and $O'X'$, $O'Y'$, the new axes. The coordinates of O', the new origin, referred to the original axes are $OL = x_0$, $LO' = y_0$. Let P be any point in the plane. The coordinates of P in the two systems are

$$OM = x, \quad MP = y; \quad \text{and} \quad O'M' = x', \quad M'P = y'.$$

The formulas required are those which express the original coordinates of P in terms of the new coordinates and the con-

stants x_0, y_0. From the figure

$$OM = OL + LM = OL + O'M',$$

$$MP = MM' + M'P = LO' + M'P,$$

hence, $$x = x' + x_0, \quad y = y' + y_0. \tag{2}$$

EXAMPLE. Transform the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ to new axes parallel respectively to the old, the new origin being at $(3, -2)$.

By (2) the formulas of transformation are

$$x = x' + 3, \qquad y = y' - 2.$$

Substituting in the given equation we have

$$(x' + 3)^2 + (y' - 2)^2 - 6(x' + 3) + 4(y' - 2) - 3 = 0,$$

which reduces to

$$x'^2 + y'^2 = 16.$$

Let the student construct the figure.

EXERCISES. 1. Determine the new equation of each of the following loci when transformed to new axes parallel respectively to the old, the new origin being taken at the point specified in each case.

(a) $x + y = 1$, $(-2, 1)$; Ans. $x' + y' - 2 = 0$.
(b) $x^2 + y^2 + 4x = 0$, $(-2, 0)$; Ans. $x'^2 + y'^2 = 4$.
(c) $4x^2 + 9y^2 + 8x + 18y - 23 = 0$, $(-1, -1)$;
(d) $4x^2 + 9y^2 + 8x + 18y - 23 = 0$, $(1, 1)$;

2. Find the coordinates of the point so situated that when the circle $x^2 + y^2 - 4x + 8y - 5 = 0$ is transformed to new axes through this point, parallel respectively to the original axes, the new equation shall be $x^2 + y^2 = 25$. Ans. $(2, -4)$.

3. Derive the general equation of a conic with the principal axis as X-axis, and the origin at the focus.

4. Derive the equation of the conic whose eccentricity is $\frac{1}{2}$, focus $(4, 0)$, and directrix the Y-axis. Ans. $3x^2 + 4y^2 - 32x + 64 = 0$.

50. The parabola.—The eccentricity, e, is 1. Therefore equation (1), p. 80, becomes after reduction

$$y^2 = 2px - p^2, \tag{3}$$

which is the equation of the parabola referred to the axes OX, OY, Fig. 43, the latter being the directrix. To obtain a simpler form of equation take the origin at the vertex V, with the tangent at V for Y-axis.

The coordinates of the new origin are $(\frac{1}{2}p, 0)$, since V bisects

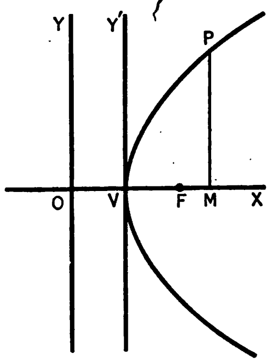


FIG. 43

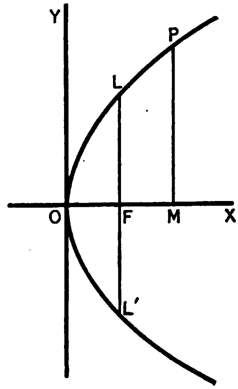


FIG. 44

OF, hence by (2) the formulas of transformation are

$$x = x' + \tfrac{1}{2}p, \qquad y = y'.$$

Substituting these values in (3) and reducing, $y'^2 = 2px'$.

Hence, dropping accents, the equation of the parabola referred to the principal axis as X-axis, and the tangent at the vertex as Y-axis, Fig. 44, is

$$y^2 = 2px. \tag{4}$$

51. Properties of the parabola.

I. *The perpendicular from any point of a parabola to the axis of the curve is the mean proportional between the latus rectum and the distance from the vertex to the foot of the perpendicular.*

This theorem is the translation of equation (4) into the language of geometry, because $y = MP$, $2p = L'L$, and $x = OM$.

II. *The parabola* $y^2 = 2px$ *lies wholly on the positive side of the* Y*-axis.*

Since p is intrinsically positive x cannot be negative in this equation without making y imaginary.

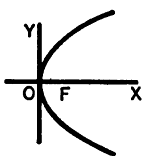
III. *Similarly* $y^2 = -2px$ *represents a parabola lying wholly on the negative side of the* Y*-axis.*

IV. *The equations* $x^2 = \pm 2py$ *represent parabolas symmetrical with respect to the* Y*-axis, with their vertices at the origin, and lying respectively above and below the* X*-axis.*

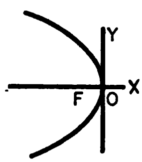
Since $2p$ is simply a positive distance the equations

$$y^2 = \pm 2px, \qquad \text{and} \qquad x^2 = \pm 2py$$

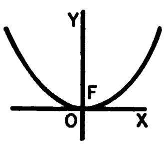
can be put in the form $y^2 = cx$ and $x^2 = cy$, where c is any constant, positive or negative, and the following figures show the different positions of the parabola according to the form of the equation and the sign of c.



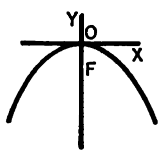
$y^2 = cx,\ c > 0$
FIG. 45(*a*)



$y^2 = cx,\ c < 0$
FIG. 45(*b*)



$x^2 = cy,\ c > 0$
FIG. 45(*c*)



$x^2 = cy,\ c < 0$
FIG. 45(*d*)

EXERCISES. 1. Make careful drawings of the following parabolas, locate the focus and directrix of each one, and draw its latus rectum.

(*a*) $y^2 = 6x$; (*b*) $x^2 = 10y$; (*c*) $y^2 = -8x$; (*d*) $x^2 = -3y$.

2. Make careful drawings of the following pairs of parabolas, and determine the coordinates of the point of intersection of each pair.

(*a*) $4y^2 = 3x$, $2x^2 = 9y$; (*b*) $y^2 + x = 0$, $x^2 + y = 0$.

3. Derive the equation of each of the parabolas determined by the following sets of conditions. The vertex in each case is at the origin.

(*a*) Passing through (4, 1), axis coincident with OX.
(*b*) Passing through $(-2, 3)$, axis coincident with OX.
(*c*) Passing through $(1, -2)$, axis coincident with OY.
(*d*) Passing through (h, k), axis coincident with OX.

Ans. (*a*) $4y^2 = x$, (*b*) $2y^2 = -9x$, (*c*) $2x^2 = -y$.

52. The ellipse.—Let OY be the directrix, and OX the principal axis of an ellipse with focus F, vertices A, A', and center C.

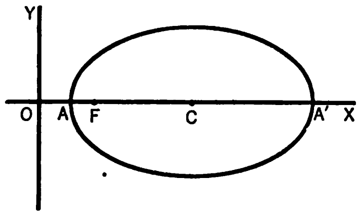


FIG. 46

I. *The distances OA, OA' from the directrix to the vertices.*

Since A, A' are on the X-axis and O is the origin, the required distances are obtained by making $y = 0$ in equation (1), p. 80, and solving for x. This gives

$$(x - p)^2 = e^2x^2, \qquad \text{or} \qquad x - p = \pm ex,$$

from which

$$x = \frac{p}{1 \pm e}.$$

Separating the two values of x, we have therefore

$$x_1 = OA = \frac{p}{1 + e}, \qquad x_2 = OA' = \frac{p}{1 - e}. \tag{5}$$

Since e is positive and less than unity it follows from (5) that x_1 and x_2 are both positive, $x_1 < x_2$, and $\frac{1}{2}p < x_1 < p$. The last statement shows that A lies nearer to F than to O.

II. *The distance AA' between the vertices.*

From the figure

$$AA' = OA' - OA$$

$\therefore$ from (5)

$$AA' = \frac{p}{1 - e} - \frac{p}{1 + e} = \frac{2ep}{1 - e^2}.$$

For convenience let $AA' = 2a$, then $a = AC = CA'$, and

$$a = \frac{ep}{1 - e^2}. \tag{6}$$

III. *The distance OC from the directrix to the center.*

From Fig. 46, and (6), p. 9,

$$OC = \tfrac{1}{2}(OA + OA') = \frac{1}{2}\left(\frac{p}{1+e} + \frac{p}{1-e}\right) = \frac{p}{1-e^2},$$

$\therefore$ from (6) above
$$OC = \frac{a}{e}. \tag{7}$$

IV. *The distance FC from the focus to the center.*

From the figure

$$FC = OC - OF = \frac{p}{1-e^2} - p = \frac{e^2 p}{1-e^2},$$

$\therefore$ from (6)
$$FC = ae. \tag{8}$$

53. The ellipse (continued).

PROBLEM. *To find the equation of the ellipse referred to rectangular axes, with the principal axis for X-axis, and the origin at the center of the ellipse.*

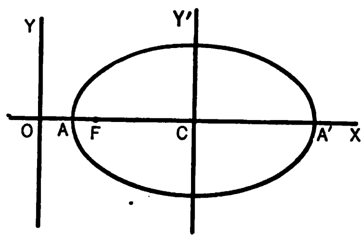


FIG. 47

This equation is conveniently derived by transforming equation (1) to the new axes here defined. The coordinates of the new origin C, referred to XOY, are $(a/e, 0)$, see (7). Hence the formulas of transformation are

$$x = x' + \frac{a}{e}, \qquad y = y'.$$

Substituting in (1), p. 80,

$$\left(x' + \frac{a}{e} - p\right)^2 + y'^2 = e^2\left(x' + \frac{a}{e}\right)^2,$$

but. $(a/e) - p = OC - OF = FC = ae$, by (7) and (8), hence the equation becomes

$$(x' + ae)^2 + y'^2 = e^2\left(x' + \frac{a}{e}\right)^2,$$

which reduces to

$$(1 - e^2)x'^2 + y'^2 = a^2(1 - e^2),$$

and finally, dividing by $a^2(1 - e^2)$, to

$$\frac{x'^2}{a^2} + \frac{y'^2}{a^2(1 - e^2)} = 1. \qquad (i)$$

Dropping accents, and writing

$$b^2 = a^2(1 - e^2), \qquad (9)$$

as $a^2(1 - e^2)$ is positive, equation (i) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (10)$$

the form in which the equation of the ellipse is most commonly used. See Fig. 48.

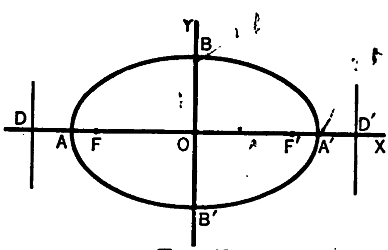


FIG. 48

If we substitute $x = 0$ in equation (10) and solve for y, the result is $y = \pm b$, showing that $OB = b$ and $OB' = -b$.

DEFINITIONS. I. *The length $AA' = 2a$ is called the* ***transverse axis****, or* ***major axis*** *of the ellipse.*

II. *The length $B'B = 2b$ is called the* ***conjugate axis****, or* ***minor axis****, of the ellipse.*

An ellipse is fully determined when its semi-major and semi-minor axes, a and b, are given. Equation (9) solved for e gives

the value of the eccentricity

$$e = \frac{\sqrt{a^2 - b^2}}{a} \tag{11}$$

in terms of a and b. Then $FO = ae$, and $DO = a/e$, are also known.

Equation (10) shows that the ellipse is symmetrical with respect to both axes, and therefore with respect to the center. Hence there is another focus F', and another directrix, through D', Fig. 48, where $OF' = FO$, and $OD' = DO$.

If in the foregoing discussion the line through the foci be taken as the Y-axis, and the perpendicular to it through the center as the X-axis, the resulting equation of the ellipse will evidently be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1. \tag{12}$$

Thus $(x^2/16) + (y^2/9) = 1$, and $(x^2/9) + (y^2/16) = 1$ both represent the same ellipse with the origin at the center, but in the first the major axis and the foci lie in the X-axis, and in the second they lie in the Y-axis.

It is customary therefore, when discussing in a general way the equation $(x^2/a^2) + (y^2/b^2) = 1$, to disregard the condition implied in definitions I and II above, that $a > b$, and to say that this equation represents an ellipse for all values of a and b which are positive and not zero. The larger of the two numbers a and b is then the length of the semi-major axis, and the other, the length of the semi-minor axis. Hence:

Theorem. *The equation $Ax^2 + By^2 = C$, where A, B, C all have the same sign and none of them is zero, always represents an ellipse with center at the origin, and axes coincident in direction with the coordinate axes.*

If this equation be divided by C it takes the form

$$\frac{Ax^2}{C} + \frac{By^2}{C} = 1, \qquad \text{or} \qquad \frac{x^2}{\frac{C}{A}} + \frac{y^2}{\frac{C}{B}} = 1,$$

which is in the form of (10) or (12), because by hypothesis C/A and C/B are both positive. If $C/A > C/B$ the major axis of the ellipse coincides with the X-axis, but if $C/A < C/B$ the major axis coincides with the Y-axis.

54. A property of the ellipse.

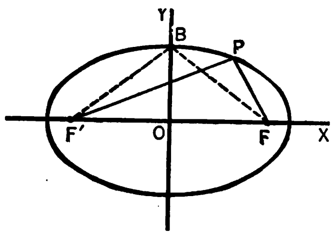


FIG. 49

DEFINITIONS. I. *A chord of a conic through a focus is called a* ***focal chord.***

II. *The distance of a point on a conic from a focus is called the* ***focal radius*** *of the point with respect to that focus.*

III. THEOREM. *The sum of the two focal radii of any point on the ellipse is constant and equal to* $2a$.

Let P be any point on the ellipse, Fig. 49. Since $P = (x, y)$, and $F = (ae, 0)$, the distance

$$FP = \sqrt{(x - ae)^2 + y^2}. \qquad (i)$$

From equation (10) $y^2 = (b^2/a^2)(a^2 - x^2)$, and from (9) $b^2/a^2 = (1 - e^2)$, hence $y^2 = (1 - e^2)(a^2 - x^2)$.

Substituting this value of y^2 in (i), we have

$$FP = \sqrt{(x - ae)^2 + (1 - e^2)(a^2 - x^2)}$$
$$= \sqrt{a^2 - 2aex + e^2x^2} = a - ex. \qquad (ii)$$

Only the positive square root is taken because the length of FP as a positive distance is desired, and $a > ex$. Similarly

$$F'P = a + ex. \qquad (iii)$$

Adding (*ii*) and (*iii*) we have

$$FP + F'P = 2a. \tag{13}$$

It follows from this that $FB = F'B = a$, which gives a convenient way of locating the foci of an ellipse when the axes are given.

EXERCISES. In exercises 1 and 2 the X- and Y-axes are to be taken coincident with the major and minor axes of the curve respectively.

1. Find the equations of the ellipses determined in each case as stated below. Make careful drawings of the curves, and locate their foci and directrices

(*a*) $a = 5,\ \ b = 3$; (*b*) $a = 4,\ \ e = \frac{1}{2}\sqrt{3}$; (*c*) $b = 2,\ \ e = \frac{3}{5}$.

Ans. (*a*) $9x^2 + 25y^2 = 225$, $e = \frac{4}{5}$, $ae = 4$, $a/e = \frac{25}{4}$.

2. Find the equations of the ellipse and of its directrices for which $a = 6$ and the foci bisect the semi-major axes.

Ans. $3x^2 + 4y^2 = 108$, $x = \pm 12$.

3. Show that in any ellipse the length of the semi-latus rectum is b^2/a.

4. For each of the following ellipses find the length of the semi-axes, the eccentricity, the coordinates of the foci, and the equations of the directrices.

(*a*) $4x^2 + 9y^2 = 36$. (*c*) $49x^2 + 25y^2 = 1225$.

(*b*) $12x^2 + 16y^2 = 192$. (*d*) $9x^2 + 5y^2 = 180$.

Ans. (*a*) $a = 3$, $b = 2$, $e = \frac{1}{3}\sqrt{5}$, $(\pm \sqrt{5}, 0)$, $x = \pm \frac{9}{5}\sqrt{5}$.

(*d*) $a = 6$, $b = 2\sqrt{5}$, $e = \frac{2}{3}$, $(0, \pm 4)$, $y = \pm 9$.

5. For each of the two ellipses $16x^2 + 25y^2 = 400$, and $9x^2 + 5y^2 = 180$, find the distance between

(*a*) center and focus, (*c*) focus and end of minor axis,

(*b*) center and directrix, (*d*) focus and nearer vertex.

Draw the curves carefully and check the distances found by measurements of the graphs.

55. The hyperbola.—Let OY be the directrix, and OX the principal axis of a hyperbola, with focus F, vertices A', A, and center C. The distances OA, OA' are found from equation (1)

in the same manner as for the ellipse,

$$x_1 = OA = \frac{p}{1+e}, \qquad x_2 = OA' = \frac{p}{1-e}. \tag{14}$$

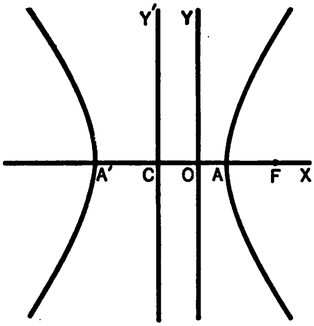


FIG. 50

Since e is positive and greater than unity, x_1 is positive and x_2 is negative, as shown in Fig. 50. Also $|x_2|^* > x_1$, and $x_1 < \frac{1}{2}p$. That is, A is nearer to the directrix than it is to the focus.

Let $A'A = 2a$ as in the ellipse. From Fig. 50

$$A'A = A'O + OA.$$

$\therefore$ using (14)

$$A'A = 2a = \frac{p}{1+e} - \frac{p}{1-e} = \frac{2ep}{e^2-1},$$

or

$$a = \frac{ep}{e^2-1}. \tag{15}$$

The student will now be able to show that for the hyperbola, as for the ellipse,

$$\textit{center to directrix} = CO = \frac{a}{e} \tag{16}$$

and

$$\textit{center to focus} = CF = ae. \tag{17}$$

PROBLEM. *To find the equation of the hyperbola referred to rectangular axes, with the principal axis as X-axis, and the origin at the center of the hyperbola.*

This equation will be derived by transformation of coordinates from equation (1), as was done in Art. 53 for the ellipse.

The coordinates of C the new origin, Fig. 50, referred to the old system are $[-(a/e), 0]$, hence the formulas of transformation

* The notation $|x_2|$ is used to indicate the absolute value of the expression between the bars, thus $|-3| = 3$.

are

$$x = x' - \frac{a}{e}, \qquad y = y'.$$

Substituting these in equation (1) we have

$$\left(x' - \frac{a}{e} - p\right)^2 + y'^2 = e^2\left(x' - \frac{a}{e}\right)^2.$$

But

$$-\frac{a}{e} - p = -CO - OF = -(CO + OF) = -CF = -ae,$$

hence $$(x' - ae)^2 + y'^2 = e^2\left(x' - \frac{a}{e}\right)^2,$$

which, after reduction, becomes

$$(e^2 - 1)x'^2 - y'^2 = a^2(e^2 - 1),$$

or, dividing by $a^2(e^2 - 1)$,

$$\frac{x'^2}{a^2} - \frac{y'^2}{a^2(e^2 - 1)} = 1. \qquad (i)$$

Dropping accents, and writing

$$b^2 = a^2(e^2 - 1), \qquad (18)$$

since $e^2 - 1$ is positive, equation (i) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (19)$$

Equation (19) shows that the hyperbola is symmetrical with respect to both $X'X$ and $Y'Y$, and hence with respect to the center O. There are therefore two foci, F, F', symmetrically situated with respect to O, and two directrices, one corresponding to each focus. The directrices cut the principal axis at points D, D', so that

$$D'O = OD.$$

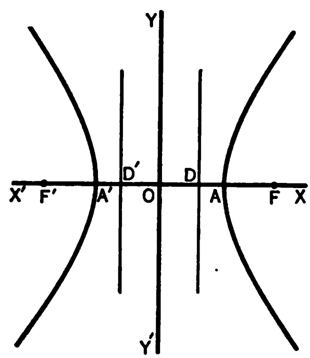


FIG. 51

A hyperbola is fully determined when the values of a and b are given. The value of the eccentricity in terms of a and b, obtained from (17) is

$$e = \frac{\sqrt{a^2 + b^2}}{a}. \qquad (20)$$

Equation (18) shows that b may be larger or smaller than a. If $e^2 > 2$, $b > a$; but if $e^2 < 2$, $b < a$.

56. Conjugate hyperbolas.—If in equation (19) the letters x and y, and a and b be interchanged respectively the result is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \qquad (21)$$

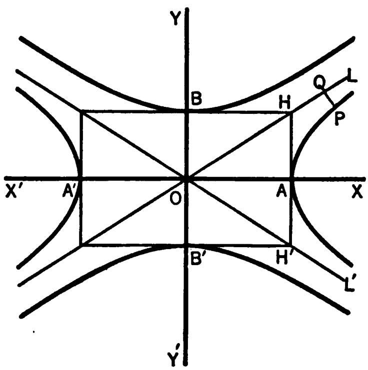


FIG. 52

This change transforms the hyperbola into one whose principal axis lies in the Y-axis, and whose vertices are at the points $(0, \pm b)$. Hence (21) is the equation of a hyperbola in this position.

The two hyperbolas are drawn in Fig. 52, where $A'A = 2a$, and $B'B = 2b$.

DEFINITIONS. I. *The two hyperbolas*

$$(A) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad (B) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

in which a and b have the same value in both equations, are called conjugate hyperbolas.

II. *The* **transverse axis** *of a hyperbola is the line joining its vertices.*

Thus, Fig. 52, $A'A = 2a$ is the transverse axis of hyperbola (A), and $B'B = 2b$ is the transverse axis of hyperbola (B).

III. *The* **conjugate axis** *of a hyperbola is the transverse axis of its conjugate hyperbola.*

Thus $B'B = 2b$ is the conjugate axis of hyperbola (A), and $A'A = 2a$ is the conjugate axis of hyperbola (B).

57. The asymptotes of a hyperbola.

THEOREM. *As x increases the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ continually approaches the lines $y = \pm (b/a)x$, but never meets them.*

The rectangle whose sides are parallel to the coordinate axes, and pass respectively through the points A, A', B, B', Fig. 52, has for its diagonals the lines $y = \pm (b/a)x$, since $AH = b$, $AH' = -b$, and $OA = a$. Considering only one of these lines, the line OL, its equation may be written in the form

$$bx - ay = 0. \tag{i}$$

Let $P(x, y)$ be in the first quadrant, on that part of the hyperbola whose vertex is at A, and draw PQ perpendicular to OL. Then by formula (20), p. 56,

$$PQ = \frac{bx - ay}{\sqrt{a^2 + b^2}}. \tag{ii}$$

From the equation of the hyperbola, $ay = \pm b \sqrt{x^2 - a^2}$. Using the positive sign for the point P, since it is in the first quadrant, we have, substituting in (ii),

$$PQ = \frac{bx - b\sqrt{x^2 - a^2}}{\sqrt{a^2 + b^2}} = \frac{b(x - \sqrt{x^2 - a^2})}{\sqrt{a^2 + b^2}} \cdot \frac{(x + \sqrt{x^2 - a^2})}{(x + \sqrt{x^2 - a^2})}$$

or

$$PQ = \frac{a^2 b}{\sqrt{a^2 + b^2}(x + \sqrt{x^2 - a^2})}. \tag{iii}$$

Thus in (iii) the length of PQ is expressed as a fraction whose numerator is constant, and whose denominator increases in-

definitely as x increases. That is PQ approaches the limit zero as x increases indefinitely.

Similarly it can be shown that in each quadrant the hyperbola approaches one of the two lines OL or OL'.

The lines to which the hyperbola is thus related are called the **asymptotes** of the curve.

In the same manner it can be shown that the conjugate hyperbola $(x^2/a^2) - (y^2/b^2) = -1$ approaches one of the lines OL or OL' in each quadrant. In other words: *The two conjugate hyperbolas $x^2/a^2 - y^2/b^2 = 1$ and $x^2/a^2 - y^2/b^2 = -1$ have for asymptotes the same pair of lines*

$$y = \pm \frac{b}{a} x. \tag{22}$$

58. A property of the hyperbola.

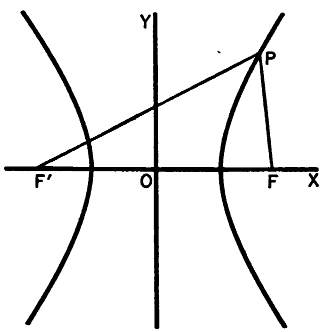


FIG. 53

THEOREM. *The difference of the focal radii of any point on the hyperbola is constant and equal to 2a.*

Let $P = (x, y)$ be any point on the curve, then we are to prove

$$F'P - FP = 2a.$$

The details of the proof which are the same as in Art. 54, III, are left to the student.

The results are

$$F'P = ex + a, \qquad FP = ex - a,$$

and hence

$$F'P - FP = 2a. \tag{23}$$

59. The equilateral or rectangular hyperbola.

DEFINITION. *If the semi-axes a and b of a hyperbola are equal the hyperbola is called an* ***equilateral*** *or* ***rectangular hyperbola.***

The equation of an equilateral hyperbola, referred to its axes as axes of coordinates, is found by putting $b = a$ in equation (19), p. 91, and is therefore

$$x^2 - y^2 = a^2. \tag{24}$$

THEOREM. *The asymptotes of an equilateral hyperbola are perpendicular to each other.*

By (22) the equations of the asymptotes to the curve of (24) are $y = x$ and $y = -x$, which are perpendicular lines.

EXERCISES. In exercises 1 and 2 the X- and Y-axes are to be taken coincident with the transverse and conjugate axes of the curve, respectively.

1. Find the equations of the hyperbolas determined in each case as stated below. Make careful drawings of the curves, and locate their foci and directrices.

(*a*) $a = 4, \quad b = 3,$ (*b*) $a = 6, \quad e = 2,$ (*c*) $b = 3, \quad e = \frac{4}{3}.$

Ans. (*a*) $9x^2 - 16y^2 = 144,\ e = \frac{5}{4},\ ae = 5,\ a/e = \frac{16}{5}.$

2. Find the equations of the hyperbola, and of its asymptotes, and the coordinates of its foci, for which $a = 6$, and the directrices bisect the semi-transverse axes. Ans. $3x^2 - y^2 = 108,\ y = \pm\sqrt{3}x,\ (\pm 12, 0).$

3. Show that in any hyperbola the length of the semi-latus rectum is b^2/a.

4. For each of the following hyperbolas find the lengths of the semi-axes, the eccentricity, the coordinates of the foci, and the equations of the directrices and asymptotes.

(*a*) $16x^2 - 9y^2 = 144,$ (*c*) $x^2 - y^2 = 4,$

(*b*) $9x^2 - 16y^2 = 144,$ (*d*) $9x^2 - 16y^2 = -144,$

Ans. (*a*) $a = 3,\ b = 4,\ e = \frac{5}{3},\ x = \pm\frac{9}{5},\ y = \pm\frac{4}{3}x.$

(*d*) $a = 3,\ b = 4,\ e = \frac{5}{3},\ y = \pm\frac{9}{5},\ y = \pm\frac{3}{4}x.$

5. Show that the four foci of two conjugate hyperbolas, and the four points of intersection of the tangents at their vertices, all lie on one circle whose center is at the common center of the two hyperbolas.

6. For each of the two conjugate hyperbolas $144x^2 - 25y^2 = \pm 4$ find the distance between

(*a*) center and focus,
(*b*) center and directrix,
(*c*) focus and asymptotes,
(*d*) focus and nearer vertex.

Check graphically, using a large scale.

60. Transformation of coordinates.

PROBLEM. *To find the formulas for transforming the equation of a locus from one pair of axes to another pair having the same origin, the original system being rectangular.*

Let XOY, Fig. 54, be the original or old system, and $X'OY'$ the new system. Let OX' make the angle θ with OX, and OY' the angle θ' with OX. Let P be any point in the plane, then the old coordinates of P are $x = OM$, $y = MP$, and the new coordinates of P are $x' = OM'$, $y' = M'P$. We must express x and y in terms of x', y' and the two angles θ and θ'.

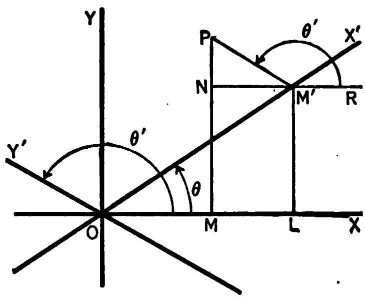


FIG. 54

Draw $M'L$ perpendicular to OX, and NR through M' parallel to OX. Then, giving attention to the directions of the different lengths,

$$\left.\begin{aligned} x &= OM = OL + M'N = x' \cos\theta + y' \cos\theta', \\ \text{and} \quad y &= MP = LM' + NP = x' \sin\theta + y' \sin\theta'. \end{aligned}\right\} \quad (25)$$

If the new system is rectangular, as well as the old, then $\theta' = 90° + \theta$, and formulas (25) take the form

$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta, \\ y &= x' \sin\theta + y' \cos\theta. \end{aligned}\right\} \quad (26)$$

When $\theta' = 90° + \theta$ the effect produced is the same as that of rotating the system of coordinates through the angle θ, because

the new X- and Y-axes make the same angle θ with the old X- and Y-axes respectively. This transformation is generally described as rotation of the axes through the angle θ.

61. The hyperbola referred to its asymptotes.

THEOREM. *The equation of a hyperbola referred to its asymptotes as axes of coordinates is $xy = c$, where c is a constant.*

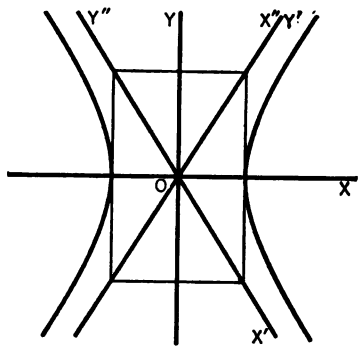


FIG. 55

To transform equation (19) p. 91 to the system of axes $X'OY'$ use (25), where

$$\theta = XOX' = -\tan^{-1}\frac{b}{a}, \qquad \text{and} \qquad \theta' = XOY' = \tan^{-1}\frac{b}{a},$$

so that by (17) p. ix

$$\begin{aligned} \sin\theta &= -\frac{b}{\sqrt{a^2+b^2}}, & \cos\theta &= \frac{a}{\sqrt{a^2+b^2}}, \\ \sin\theta' &= \frac{b}{\sqrt{a^2+b^2}}, & \cos\theta' &= \frac{a}{\sqrt{a^2+b^2}}, \end{aligned} \tag{i}$$

and hence, substituting in (25)

$$x = \frac{ax' + ay'}{\sqrt{a^2+b^2}}, \qquad y = \frac{-bx' + by'}{\sqrt{a^2+b^2}}. \tag{ii}$$

Finally, substituting these values of x and y in (19), we have

$$\frac{(x'+y')^2}{a^2+b^2} - \frac{(-x'+y')^2}{a^2+b^2} = 1$$

or, reducing,

$$x'y' = \tfrac{1}{4}(a^2+b^2). \tag{iii}$$

If OX'', OY'', Fig. 55, be taken as the new X- and Y-axes

8

respectively, instead of OX' and OY', the values of $\sin\theta$, $\cos\theta$, $\sin\theta'$, $\cos\theta'$ will be the same as before, except that $\sin\theta$ will be positive and $\cos\theta'$ negative. Hence instead of (*ii*) we have

$$x = \frac{ax' - ay'}{\sqrt{a^2 + b^2}}, \qquad y = \frac{bx' + by'}{\sqrt{a^2 + b^2}},$$

and substituting in (19) and reducing the result is

$$x'y' = -\tfrac{1}{4}(a^2 + b^2). \tag{iv}$$

The form of (*iii*) and (*iv*) shows that the product of the coordinates of any point on the hyperbola, referred to the asymptotes, is a constant. Dropping the accents, and replacing $\pm\frac{1}{4}(a^2 + b^2)$ by c, a constant, the equation takes the form

$$xy = c. \tag{27}$$

If c is positive the signs of x and y must be alike, so that the curve lies in the first and third quadrants, but if c is negative the signs of x and y are unlike, and the curve lies in the second and fourth quadrants.

If the coordinate axes (the asymptotes) are at right angles the hyperbola is equilateral.

The equilateral hyperbola is the form of this curve which most commonly occurs in problems of physics and engineering, and the equation referred to the asymptotes as axes of coordinates is the one most often employed.

EXERCISE. Make careful drawings of the two hyperbolas $xy = 4$ and $xy = -4$, referred to rectangular axes.

62. Equations of the conics in more general form.

I. *Equation of the parabola with vertex at (x_0, y_0), and axis parallel to one of the coordinate axes.*

Let V be the vertex of a parabola, whose axis VX' is parallel to OX. Referred to $X'VY'$ as coordinate axes, with x', y' as

variables, the equation of the parabola will be

$$y'^2 = 2px'. \tag{i}$$

Let the coordinates of V, referred to XOY, be $OM = x_0$, $MV = y_0$, then the coordinates of O, referred to $X'VY'$, will be $MO = -x_0$, $VM = -y_0$, and hence by (2), p. 81, the formulas for transforming from the system $X'VY'$ to the system XOY will be

$$x' = x - x_0, \quad y' = y - y_0. \tag{ii}$$

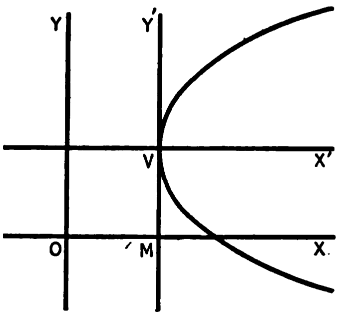


FIG. 56

Substituting these values in (i) we have

$$(y - y_0)^2 = 2p(x - x_0), \tag{28}$$

which is the required equation.

In more general form (28) may be written

$$(y - y_0)^2 = c(x - x_0), \tag{29}$$

where c is any constant positive or negative, and the focus of the parabola will lie on the positive or negative side of V according as c is positive or negative.

Similarly

$$(x - x_0)^2 = c(y - y_0) \tag{30}$$

is the equation of a parabola with vertex at (x_0, y_0) and axis parallel to the Y-axis. Compare Art. 51, IV.

II. *Equations of the ellipse and hyperbola with center at* (x_0, y_0).

If the coordinates of C, Fig. 57, referred to the system XOY, are (x_0, y_0), then as in I, by transformation of equation (10), p. 86, from the axes $X'CY'$ to the system XOY, the equation of the ellipse with center at (x_0, y_0), and axes parallel to the coordinate

axes is found to be

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1. \tag{31}$$

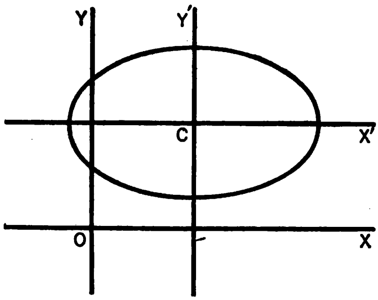


FIG. 57

In connection with equation (31) refer to the latter portion of Art. 53.

Similarly by transformation of equations (19), p. 91, and (21), p. 92, the equations of the hyperbolas with center at (x_0, y_0), and axes parallel to the coordinate axes are found to be

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = \pm 1. \tag{32}$$

In like manner the equation of the hyperbola with center at (x_0, y_0), and axes parallel to the asymptotes is derived from (27).

$$(x - x_0)(y - y_0) = c. \tag{33}$$

63. The parabolic arch.—Let OVA, Fig. 58, represent the outline of an arch in the form of a parabola, and let the distance OA, called the span, be represented by s, and the height MV, called the rise, by r. Taking the axes as indicated in the figure, the vertex of the parabola is at the point $(\frac{1}{2}s, r)$ and its axis is parallel to OY. Hence from (30) the equation will be

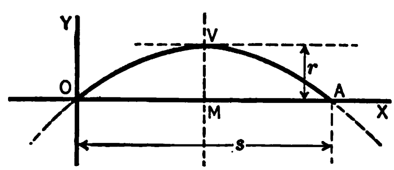


FIG. 58

$$(x - \tfrac{1}{2}s)^2 = c(y - r), \tag{i}$$

in which c is to be determined so that the curve shall pass through the origin. Substituting $x = 0$, $y = 0$ in (i), we have

$$\tfrac{1}{4}s^2 = -cr, \qquad \text{or} \qquad c = -\frac{s^2}{4r}.$$

Substituting this value of c in (i) and reducing

$$4rx^2 - 4rsx + s^2y = 0. \tag{34}$$

64. The equation $Ax^2 + By^2 + Gx + Fy + C = 0$. This equation, except in certain cases easily recognized, always represents a conic. It will now be shown, by means of examples, how to treat an equation of this form in order to determine the nature and position of the locus to which it corresponds.

1. $2x^2 + 5y^2 - 8x + 7y - 12 = 0.$

By rearranging the terms the equation becomes

$$2(x^2 - 4x) + 5(y^2 + \tfrac{7}{5}y) = 12,$$

and completing the squares in the two parentheses

$$2(x^2 - 4x + 4) + 5(y^2 + \tfrac{7}{5}y + \tfrac{49}{100}) = 12 + 8 + \tfrac{49}{20},$$

or

$$2(x - 2)^2 + 5(y + \tfrac{7}{10})^2 = \tfrac{449}{20},$$

which reduces to

$$\frac{(x-2)^2}{\frac{449}{40}} + \frac{(y + \frac{7}{10})^2}{\frac{449}{100}} = 1.$$

Comparing this with (31) it is seen that the figure is an ellipse with center at $(2, -\tfrac{7}{10})$, $a = \sqrt{\tfrac{449}{40}}$, and $b = \sqrt{\tfrac{449}{100}}$. The major axis is parallel to OX.

2. $y^2 + 6x - 7y + 15 = 0.$

The term in x^2 is missing, and the operation is therefore modified accordingly. We write first

$$y^2 - 7y = -6x - 15,$$

and then

$$y^2 - 7y + \tfrac{49}{4} = -6x - 15 + \tfrac{49}{4},$$

or

$$(y - \tfrac{7}{2})^2 = -6(x + \tfrac{11}{24}).$$

By comparison with (29) this is seen to be a parabola with vertex at $(-\tfrac{11}{24}, \tfrac{7}{2})$, and axis parallel to OX. The focus is $1\tfrac{1}{2}$ units on the negative side of the vertex, and the length of the latus rectum is 6. The student should construct the curves in the two preceding examples.

The examples which follow illustrate exceptional cases which arise.

3. $2x^2 + y^2 - 8x + 4y + 12 = 0.$

This equation reduces to $2(x - 2)^2 + (y + 2)^2 = 0$. Hence the only real coordinates which will satisfy it are $x = 2, y = -2$. It therefore represents the point to which an ellipse shrinks as its semi-axes a and b simultaneously approach zero.

Since the equation can be written in the form

$$[\sqrt{2}(x - 2) + i(y + 2)][\sqrt{2}(x - 2) - i(y + 2)] = 0$$

it may also be interpreted as representing a pair of imaginary lines meeting in the real point $(2, -2)$. See footnote, p. 67.

4. $2x^2 - y^2 - 8x + 4y + 4 = 0.$

This equation reduces to $2(x - 2)^2 - (y - 2)^2 = 0$. It therefore represents the two straight lines

$$\sqrt{2}(x - 2) + (y - 2) = 0, \qquad \sqrt{2}(x - 2) - (y - 2) = 0.$$

This is a limiting form of the hyperbola as its semi-axes simultaneously approach zero.

5. $2x^2 + y^2 - 8x + 4y + 16 = 0.$

This equation after transformation takes the form

$$\frac{(x - 2)^2}{2} + \frac{(y + 2)^2}{4} = -1,$$

which is not satisfied by any real values of x and y. It is therefore the equation of an imaginary locus.

Exercises on Chapter V

Normal Exercises

1. Construct accurately the conics determined as follows

(a) $e = 1$, distance from focus to directrix $= \frac{3}{4}$ inch.

(b) $e = \frac{2}{3}$, distance from focus to directrix $= 1\frac{1}{4}$ inches.

(c) $e = \frac{3}{2}$, distance from focus to directrix $= 1$ inch.

2. Derive the equations of the conics in Ex. 1, using the directrix as Y-axis, and the principal axis as X-axis.

3. Find the equations of the conics determined by the following sets of conditions, and draw each curve.

(a) The parabola, with vertex at the origin, $p = 2$, and focus on the positive extension of the X-axis.

(b) The parabola, with vertex at the origin, and focus at $(0, -4)$.

(c) The ellipse, with center at the origin, $a = 4$, $b = 2$, and major axis in the X-axis.

(d) The ellipse, with center at the origin, one focus at $(0, 3)$, and $e = \frac{3}{4}$.

(e) The hyperbola, with center at the origin, $a = 2$, $b = 4$, and transverse axis in the X-axis.

(f) The hyperbola, with center at the origin, $a = 2$, $e = \frac{4}{3}$, and transverse axis in the Y-axis.

(g) The hyperbola, with $a = b = 4$, referred to its asymptotes as axes of coordinates.

4. Find the equations of the conics determined by the following sets of conditions, and draw each curve.

(a) The parabola with vertex at $(-2, 4)$, $p = 4$, the axis parallel to the X-axis, and the curve lying on the positive side of the vertex.

Ans. $y^2 - 8x - 8y = 0$.

(b) The parabola passing through $(0, 4)$, with vertex at $(2, -3)$, and axis parallel to the X-axis. Ans. $2y^2 + 49x + 12y - 80 = 0$.

(c) The ellipse, with center at $(2, 1)$, $a = 4$, $b = \frac{2}{3}$, and major axis parallel to the X-axis.

(d) The ellipse passing through the origin, with center at $(-2, -3)$, one vertex at $(-2, 7)$, and major axis parallel to the Y-axis.

Ans. $91x^2 + 4y^2 + 364x + 24y = 0$.

(*e*) **The hyperbola, with center at $(-1, -1)$, $a = 3$, $b = 7$, and transverse axis parallel to the Y-axis.**

(*f*) **The hyperbola passing through $(8, 0)$, with center at $(4, 2)$, $b = 2$, and transverse axis parallel to the X-axis.**

Ans. $x^2 - 2y^2 - 8x + 8y = 0$.

5. Determine the nature and position of the conics having the equations given below. If the curve is an *ellipse* or *hyperbola* find its *eccentricity* and *semi-axes*, the *coordinates* of its *center* and *foci*, and the *equations* of its *axes* and *directrices*. If the curve is a *parabola* find the *coordinates* of its *vertex* and *focus*, and the *equations* of its *axis* and *directrix*.

(*a*) $y^2 = -10x$,	(*f*) $4x^2 - y^2 = -9$,
(*b*) $9x^2 = 4y$,	(*g*) $4x^2 - y^2 - 12x - 6y + 23 = 0$,
(*c*) $x^2 + 9y^2 = 9$,	(*h*) $9x^2 + 4y^2 + 18x - 27 = 0$,
(*d*) $4x^2 + y^2 = 9$,	(*i*) $4y^2 - 7x - 8y + 1 = 0$,
(*e*) $x^2 - 9y^2 = 9$,	(*k*) $3x^2 + 2y^2 + 12x - 12y + 34 = 0$.

6. Find the lengths of the focal radii of those points on the ellipse $7x^2 + 16y^2 = 28$ whose abscissa is 1. Ans. $\frac{5}{4}$, $\frac{11}{4}$.

7. Find the lengths of the focal radii of those points on the hyperbola $5x^2 - 4y^2 = 5$ whose abscissa is -3. Ans. $\frac{7}{2}$, $\frac{11}{2}$.

8. Write the equations of the asymptotes of the following hyperbolas. Construct each curve and its asymptotes.

(*a*) $4x^2 - 4y^2 = 9$,	(*c*) $25x^2 - 4y^2 = 49$,
(*b*) $36x^2 - 9y^2 = -16$,	(*d*) $9x^2 - 36y^2 = -16$.

9. Find the equations of the hyperbolas in Ex. 8 referred to their asymptotes as axes of coordinates.

10. (*a*) Transform $x^2 + 4y = 8$ to new axes parallel respectively to the old axes, with the new origin at $(2, 4)$.

Ans. $x'^2 + 4x' + 4y' + 12 = 0$.

(*b*) Transform $5x^2 + 4y^2 - 20x + 8y + 23 = 0$ in a similar manner, with the new origin at $(2, -1)$. Ans. $5x'^2 + 4y'^2 = 1$.

(*c*) Transform $x^2 - y^2 = 4$ to new axes through the same origin, the new axes making angles of 30° and $-30°$ respectively with the original X-axis. Ans. $x'^2 + 4x'y' + y'^2 = 8$.

(*d*) Transform $x^2 + 4y^2 + 4xy - 36 = 0$ in a similar manner to new axes having the slopes 2 and $-\frac{1}{2}$ respectively.

Ans. $5x'^2 - 36 = 0$.

General Exercises

NOTE. In exercises 11–19 the equations of the curves are to be found from the data, and the results checked graphically.

11. The parabola with vertex at $(-4, 0)$, and directrix $x = 2$.
Ans. $y^2 - 24x - 96 = 0$.

12. The parabola with axis parallel to the Y-axis, which passes through the points $(0, 0)$, $(2, 4)$, $(-2, 4)$. Ans. $x^2 - y = 0$.

13. The parabola with axis parallel to the X-axis, which passes through the points $(2, 4)$, $(4, 8)$, $(1, -1)$.
Ans. $y^2 + 3y - 30x + 32 = 0$.

14. The ellipse which has the line from $(2, 4)$ to $(8, 4)$ for major axis, and one focus at $(4, 4)$. Ans. $8x^2 + 9y^2 - 80x - 72y + 272 = 0$.

15. The ellipse with center at $(10, -5)$, one focus at $(4, -5)$, and $x = 2$ for the corresponding directrix.
Ans. $x^2 + 4y^2 - 20x + 40y + 152 = 0$.

16. The ellipse with one focus at $(4, 2)$, one vertex at $(4, -10)$, and the minor axis in the line $y = -2$.
Ans. $4x^2 + 3y^2 - 32x + 12y - 116 = 0$.

17. The hyperbola with center at $(0, 0)$, one focus at $(0, 6)$, and one vertex at $(0, 4)$. Ans. $5y^2 - 4x^2 = 80$.

18. The hyperbola whose transverse and conjugate axes are in the X- and Y-axes respectively, whose eccentricity is 2, and which passes through the point $(2, 3)$.

19. The hyperbola with transverse axis in the Y-axis, which has $3y - 2x = 0$ for one asymptote, and one focus in the line $y = -4$.
Ans. $117y^2 - 52x^2 = 576$.

20. Regarding the circle as an ellipse in which $b = a$, locate its foci and directrices, and determine its eccentricity.

21. Show that the equation of an ellipse referred to its major axis as X-axis, and the tangent at the left-hand vertex as Y-axis is

$$y^2 = \frac{b^2}{a^2}(2ax - x^2), \qquad \text{or} \qquad y^2 = (1 - e^2)(2ax - x^2).$$

22. The asymptotes of the hyperbola $xy = 4$ are inclined to each other at an angle of 60°. Find a and b. Ans. $2\sqrt{3}$, 2.

23. Using the definition of Art. 45 find the equation of

(a) the parabola with focus at $(2, 4)$ and $x + y = 1$ for directrix,
Ans. $x^2 - 2xy + y^2 - 6x - 14y + 39 = 0$.

(*b*) the ellipse with focus at $(-2, 2)$, eccentricity $= \frac{1}{2}$, and $3x - 2y + 4 = 0$ for directrix,

Ans. $43x^2 + 12xy + 48y^2 + 184x - 192y + 400 = 0.$

(*c*) the hyperbola with focus at $(-1, -2)$, eccentricity $= \frac{3}{2}$, and $4x - y = 0$ for directrix.

Ans. $76x^2 - 72xy - 59y^2 - 136x - 272y - 340 = 0.$

24. Derive an equation which will represent all ellipses whose foci are the points $(4, 0)$ and $(-4, 0)$. Ans. $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - 16} = 1,\ a > 4.$

25. Derive an equation which will represent all hyperbolas whose foci are the points $(4, 0)$ and $(-4, 0)$. Ans. $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - 16} = 1,\ a < 4.$

26. Find for what values of c the straight line $3x + 4y = c$ cuts the circle $x^2 + y^2 = 25$ in (*a*) two real and distinct points, (*b*) two coincident points, (*c*) two imaginary points.

Ans. (*a*) $-25 < c < 25$, (*b*) $c = 25$, (*c*) $c^2 > 625$.

27. Find the coordinates of the real intersections of the following pairs of curves. Draw the figure for each case.

(*a*) $x^2 + y^2 = 16$, and $y^2 = 6x$. Ans. $(2, \pm 2\sqrt{3})$.

(*b*) $x^2 - y^2 = 10$, and $xy = 12$. Ans. $(3\sqrt{2}, 2\sqrt{2})$, $(-3\sqrt{2}, -2\sqrt{2})$.

(*c*) $x^2 + 6y^2 = 1$, and $x^2 + 5y = 0$.

(*d*) $9x^2 + 25y^2 = 225$, and $x^2 - 16y^2 = 16$.

28. Find the coordinates of the intersections of the parabola $y^2 = 3x$ with the curve $4y^2 = x(2x - 3)(2x - 7)$. Check graphically.

Ans. $(0, 0)$, $(\frac{1}{2}, \pm \frac{1}{2}\sqrt{6})$, $(, \frac{9}{2} \pm \frac{3}{2}\sqrt{6})$.

29. Find the coordinates of the real points where the ellipse $16x^2 + y^2 = 64$, and the curve $y^2 = x(x - 2)(x - 4)$ intersect. Check the results graphically. Ans. $(2, 0)$.

CHAPTER VI

TANGENTS, NORMALS, DIAMETERS, POLES AND POLARS

65. Definition of a tangent.

I. *If a secant intersects a curve in two points, one of which is fixed, and then moves in such a way as ultimately to bring these two points into coincidence, this limiting position of the secant is called the* **tangent** *to the curve at the fixed point.*

Thus if the secant AB_1, Fig. 59 (a), moves so that the point A

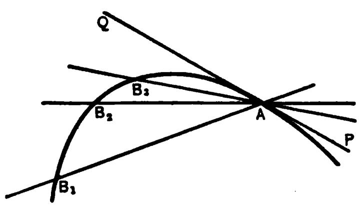


FIG. 59(a)

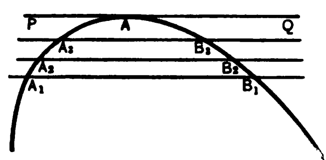


FIG. 59(b)

remains fixed while B_1 travels on the curve through the positions B_2, B_3, etc., until it comes into coincidence with A, then PQ, the limiting position of the secant, is the tangent to the curve at A. A secant may be moved into the position of tangency in other ways. Thus in Fig. 59 (b) it moves so as to remain parallel to its original position, until its two points of intersection come into coincidence and the secant becomes the tangent A.

This explanation shows that a tangent to a curve may be regarded as meeting the curve at the point of tangency in two coincident points.

66. Tangents to the conics* in terms of the slope. The question to be investigated may be stated thus:

Given the equation of a conic to derive the equation of a straight line, with given slope, which shall be tangent to the curve.

I. *The tangent to the ellipse* $(x^2/a^2) + (y^2/b^2) = 1$ *in terms of the slope.*

Let AB be a straight line having the given slope m. Its equation will then be

$$y = mx + k. \tag{i}$$

If y be eliminated between this equation and that of the ellipse the result is

$$\frac{x^2}{a^2} + \frac{(mx + k)^2}{b^2} = 1,$$

or, after reduction,

$$(a^2m^2 + b^2)x^2 + 2a^2mkx + a^2k^2 - a^2b^2 = 0. \tag{ii}$$

The roots of this quadratic in x are the abscissas MP, NQ of the points where AB cuts the ellipse. If AB be moved so as

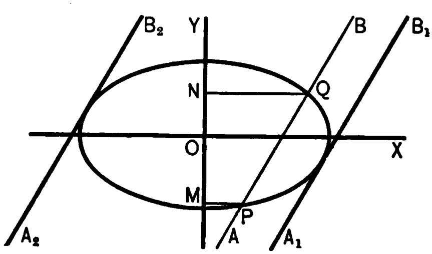


FIG. 60

to remain parallel to its original position, m will remain constant in (i) and (ii), but k will vary, and when AB takes either of the positions A_1B_1, A_2B_2, tangent to the ellipse, the roots of equa-

* The word "conic" as used here includes the circle, as well as the ellipse, hyperbola, and parabola.

tion (*ii*) will become equal. When these roots are equal (see II (*b*) p. vii)

$$4a^4m^2k^2 - 4(a^2m^2 + b^2)(a^2k^2 - a^2b^2) = 0,$$

or

$$b^2k^2 - a^2m^2b^2 - b^4 = 0,$$

from which

$$k = \pm \sqrt{a^2m^2 + b^2}.$$

These are therefore the values of k for which the straight line $y = mx + k$ will be tangent to the ellipse, and hence these tangents are

$$y = mx \pm \sqrt{a^2m^2 + b^2}. \qquad (1)$$

It may be observed that the first form of this result corresponds to the tangent A_2B_2, Fig. 60, whose y-intercept is positive, and the second form corresponds to A_1B_1, whose y-intercept is negative.

II. *The equation of the tangent to the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

in terms of the slope m, is

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \qquad (2)$$

This result is obtained simply by changing the sign of b^2 throughout the foregoing demonstration.

III. *The equation of the tangent to the circle* $x^2 + y^2 = r^2$ *in terms of the slope m is*

$$y = mx \pm r\sqrt{m^2 + 1}. \qquad (3)$$

This result is obtained from (1) by substituting $a = b = r$.

The student should for practice derive (2) and (3) independently, as (1) is derived.

IV. *The equation of the tangent to the parabola* $y^2 = 2px$ *in terms of the slope.*

Let $y = mx + k$ be a straight line having the given slope.

As in I eliminate y between the equation of the line and that of the parabola, then

$$m^2x^2 + 2(mk - p)x + k^2 = 0.$$

This equation will have equal roots if

$$4(mk - p)^2 - 4m^2k^2 = 0, \quad \text{or} \quad -2mpk + p^2 = 0,$$

from which

$$k = \frac{p}{2m}.$$

Hence the required equation is

$$y = mx + \frac{p}{2m}. \tag{4}$$

Since only one value of k is found, there is in the case of the parabola only one tangent having any given slope.

Exercises. The complete solution of the following exercises includes in every case a well-drawn figure.

1. Write the equation of the tangents

(a) to $x^2 + y^2 = 25$, having the slope $+\sqrt{3}$;
(b) to $(x^2/9) - (y^2/25) = 1$, having the slope -2;
(c) to $y^2 = 12x$, having the slope 3;
(d) to $(x^2/16) + (y^2/9) = 1$, having the slope -1;
(e) to $(x^2/5) + (y^2/2) = 1$, having the slope $\frac{2}{3}$;
(f) to $(x^2/36) - (y^2/25) = 1$, having the slope 1.

Ans. (a) $y = \sqrt{3}x \pm 10$, (b) $y = -2x \pm \sqrt{11}$, (c) $y = 3x + 1$.

2. Show that the equations of the tangents to $(x^2/b^2) + (y^2/a^2) = 1$, with slope m, are $y = mx \pm \sqrt{a^2 + b^2m^2}$; and write the equations of the tangents to $7x^2 + 3y^2 = 21$ having the slope $-\sqrt{3}$.

Ans. $y = -\sqrt{3}x \pm 4$.

3. Show that the equation of the tangent to $x^2 = 2py$, having the slope m, is $y = mx - \frac{1}{2}pm^2$; and write the equation of the tangent to $x^2 = 10y$ having the slope -2.

4. Show that the equations of the tangents to $xy = c$ in terms of the slope m are $y = mx \pm 2\sqrt{-mc}$; and write the equation of the tangent to $xy = 3$, having the slope $-\frac{3}{4}$.

5. Show by transformation of coordinates that the equations of the tangents to the circle $(x - h)^2 + (y - k)^2 = r^2$ with the slope m are $y - k = m(x - h) \pm r\sqrt{1 + m^2}$.

6. Deduce from the result of Ex. 5 the equations of the tangents to $x^2 + y^2 - 4x + 2y - 11 = 0$ having the slope $\frac{2}{3}$. Also solve the problem independently.

Ans. $2x - 3y - 7 \pm 4\sqrt{13} = 0$.

7. Derive the equation of the tangent to $x^2 - 10x - 8y + 41 = 0$ having the slope 2. Ans. $y = 2x - 16$.

8. Derive the equation of the line with slope $-\frac{1}{2}$ tangent to $4y^2 - 16x + 4y + 17 = 0$. Ans. $x + 2y + 4 = 0$.

9. Deduce from equation (2), and from the results of Ex. 4 the values of m for which the tangents will be imaginary in each case, and draw figures illustrating the results.

67. Tangents to conics in terms of the coordinates of the points of tangency.

I. PROBLEM. *To derive the equation of the tangent to the conic*

$$Ax^2 + By^2 + Gx + Fy + C = 0 \qquad (i)$$

at the point (x_1, y_1).

Let AB be a portion of the conic whose equation is (i), $P_1(x_1, y_1)$ the point of tangency, and $P_2(x_1 + h, y_1 + k)$ any other point on the conic, then $P_1N = h$, and $NP_2 = k$. The slope of the chord P_1P_2 is $\tan NP_1P_2 = k/h$.

As P_2 moves on the curve so as to approach P_1, the chord P_1P_2 approaches the tangent P_1Q, and at the same time k and h approach the limit zero. Hence the slope of the tangent P_1Q is the limit of the ratio k/h as k and h approach the limit zero. The value of this limit may be determined as follows.

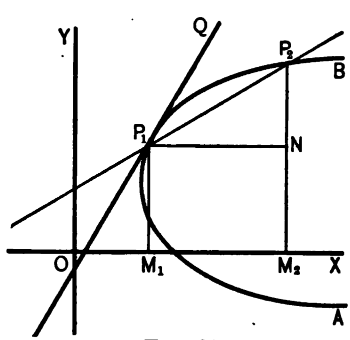


FIG. 61

Since P_1 and P_2 are both on the conic their coordinates will satisfy equation (*i*), hence

$$Ax_1^2 + By_1^2 + Gx_1 + Fy_1 + C = 0, \qquad (ii)$$

and

$$A(x_1 + h)^2 + B(y_1 + k)^2 + G(x_1 + h) + F(y_1 + k) + C = 0. \qquad (iii)$$

Expanding (*iii*) and subtracting (*ii*) from it the result is

$$2Ax_1h + Ah^2 + 2By_1k + Bk^2 + Gh + Fk = 0,$$

or

$$h(2Ax_1 + Ah + G) + k(2By_1 + Bk + F) = 0,$$

from which

$$\frac{k}{h} = -\frac{2Ax_1 + Ah + G}{2By_1 + Bk + F}. \qquad (iv)$$

Hence from (*iv*), as h and k approach the limit zero,

$$\text{Lim}\,\frac{k}{h} = -\frac{2Ax_1 + G}{2By_1 + F},$$

which is the required slope of the tangent at (x_1, y_1). By equation (5), p. 41, the equation of the tangent is therefore

$$y - y_1 = -\frac{2Ax_1 + G}{2By_1 + F}(x - x_1). \qquad (v)$$

Clearing this equation of fractions and rearranging the terms it becomes

$$2Ax_1x + 2By_1y + Gx - Gx_1 + Fy - Fy_1 - 2Ax_1^2 - 2By_1^2 = 0. \qquad (vi)$$

Adding to (*vi*) twice equation (*ii*), the result is

$$2Ax_1x + 2By_1y + Gx + Gx_1 + Fy + Fy_1 + 2C = 0,$$

and dividing by 2, we have finally

$$Ax_1x + By_1y + \tfrac{1}{2}G(x + x_1) + \tfrac{1}{2}F(y + y_1) + C = 0, \qquad (5)$$

which is in more convenient form than (*v*) or (*vi*), and is the required result.

II. By comparing (5) with the equation of the conic (*i*) it will be seen that the two equations have well-defined resemblances. It is easy therefore, when the equation of a conic is given, to write the equation of the tangent at a given point. The following example shows how this can be done.

Let the conic be

$$2x^2 - 3y^2 + 7x - 4y - 19 = 0.$$

Rewrite the equation thus

$$2xx - 3yy + \tfrac{7}{2}(x + x) - 2(y + y) - 19 = 0,$$

and affix subscripts to *one of the variables* in each term where they are multiplied, and in each pair of terms in parenthesis, thus obtaining

$$2x_1x - 3y_1x + \tfrac{7}{2}(x + x_1) - 2(y + y_1) - 19 = 0,$$

which is the equation of the tangent at the point (x_1, y_1). If the point (3, 2) be taken as the point of tangency, the equation of the tangent to the conic at this point will be

$$6x - 6y + \tfrac{7}{2}(x + 3) - 2(y + 2) - 19 = 0,$$

or

$$19x - 16y - 25 = 0.$$

III. From equation (5) it is easy to write the equations of the tangents to the conics in the following special cases by substituting the appropriate values of A, B, G, F, C. Thus the equation of the tangent at the point (x_1, y_1) to the

circle	$x^2 + y^2 = r^2$	is	$x_1x + y_1y = r^2.$	(6)
parabola	$y^2 = 2px$	is	$y_1y = p(x + x_1).$	(7)
ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	is	$\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1.$	(8)
hyperbolas	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \pm 1$	is	$\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = \pm 1.$	(9)

The student should derive each of these equations independently by the process used in deriving (5).

IV. To derive the equation of the tangent at the point (x_1, y_1) to the hyperbola $xy = c$, with either rectangular or oblique axes,

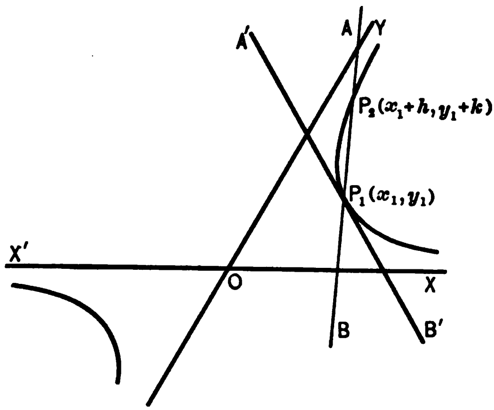


FIG. 62

take two points on the curve, $P_1(x_1, y_1)$, $P_2(x_1 + h, y_1 + k)$, and draw the secant AB, through them. Its equation by (2), p. 40, and Art. 29, will be

$$y - y_1 = \frac{k}{h}(x - x_1). \tag{i}$$

As P_2 approaches P_1 on the curve the secant P_1P_2 approaches the tangent at P_1, and h and k approach the limit zero. To find the limit of k/h substitute the coordinates of P_1 and P_2 in the equation $xy = c$, obtaining

$$x_1y_1 = c, \qquad \text{and} \qquad (x_1 + h)(y_1 + k) = c.$$

Expanding the second of these and subtracting the first from it the result is

$$kx_1 + hy_1 + hk = 0,$$

from which

$$\frac{k}{h} = -\frac{y_1 + k}{x_1}.$$

Hence as h and k approach the limit zero,

$$\text{Lim}\,\frac{k}{h} = -\frac{y_1}{x_1},$$

and therefore, substituting in (i), we have

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1).$$

Reducing, and remembering that $x_1y_1 = c$, the result is

$$y_1x + x_1y = 2c. \qquad (10)$$

68. The normal at any point on a conic.

DEFINITION. *The perpendicular to the tangent to any curve at the point of tangency is called the* **normal** *to the curve at that point.*

From the results given in (6), (7), (8), (9), p. 113, and from equation (16), p. 52, the student can easily verify that *the equation of the normal at* (x_1, y_1) *to*

$$x^2 + y^2 = r^2 \quad \text{is} \quad y - y_1 = \frac{y_1}{x_1}(x - x_1) \quad \text{or} \quad y = \frac{y_1}{x_1}x. \qquad (11)$$

$$y^2 = 2px \quad \text{is} \quad y - y_1 = -\frac{y_1}{p}(x - x_1). \qquad (12)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1). \qquad (13)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1 \quad \text{is} \quad y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1). \qquad (14)$$

Note that (11) shows the normal at any point of the circle to be a line through the center, in agreement with a well-known property of the circle.

EXERCISES. The complete solution of the following exercises includes in every case a well-drawn figure.

1. Write the equations of the tangent and normal

(a) to $x^2 + y^2 = 25$ at the point $(-3, 4)$,
(b) to $y^2 = 8x$ at the point $(2, -4)$,
(c) to $4x^2 + 9y^2 = 25$ at the point $(2, 1)$,
(d) to $x^2 - y^2 = 16$ at the point $(-5, 3)$,
(e) to $xy = 8$ at the point $(4, 2)$,

Ans. (a) Tan. $3x - 4y + 25 = 0$, Nor. $4x + 3y = 0$.
(b) Tan. $x + y + 2 = 0$, Nor. $x - y - 6 = 0$.
(c) Tan. $8x + 9y - 25 = 0$, Nor. $9x - 8y - 10 = 0$.

2. The point $(-1, 2)$ is on each of the following curves. Find the equations of the tangent and normal to each curve at that point.

(a) $4x^2 + y^2 - 24x + 4y - 40 = 0$, Ans. Tan. $4x - y + 6 = 0$, Nor. $x + 4y - 7 = 0$.
(b) $4x^2 - y^2 - 24x + 4y - 32 = 0$, Ans. Tan. $x + 1 = 0$, Nor. $y - 2 = 0$.
(c) $4x^2 - 24x + 4y - 36 = 0$,

3. Derive the equations of the tangents to the conics in Ex. 1, at the points indicated for each, by using the process of Art. 67, I.

4. Derive the equation of the tangent to the curve $y = x^3$, at the point $(2, 8)$, by the method of Art. 67, I. Ans. $y = 12x - 16$.

5. Derive the equations of the tangents to the curves $x^2 + y^2 = 25$ and $3y^2 = 16x$ at one of their points of intersection, and determine the angle between the tangents. Ans. $\tan^{-1} \frac{17}{6}$.

6. Using equation (6) determine the equations of the two tangents to the circle $x^2 + y^2 = 25$ which pass through the point $(1, 7)$.

Ans. $4x + 3y = 25$, $-3x + 4y = 25$.

7. Using equation (6), show by transformation of coordinates that the equation of the tangent to $(x - h)^2 + (y - k)^2 = r^2$, at the point (x_1, y_1) is

$$(x - h)(x_1 - h) + (y - k)(y_1 - k) = r^2.$$

69. Diameters of conics.

I. DEFINITION. *A **diameter** of a conic is the locus of the middle points of a system of parallel chords.*

The definition of the diameter of a circle in elementary geometry, a line passing through the center, is included in the definition just given, because a diameter of a circle bisects all chords perpendicular to it.

II. Theorem. *A diameter of a parabola is a straight line parallel to the axis of the curve,* and conversely, *every line parallel to the axis of a parabola is a diameter.*

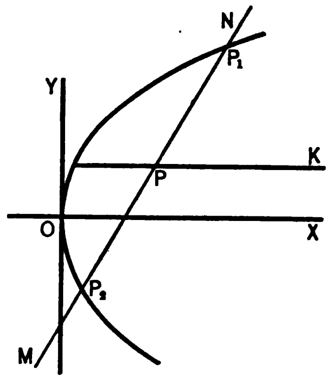


Fig. 63

Let MN be a line with slope m, meeting the parabola $y^2 = 2px$ in the points P_1, P_2. The equation of MN may be written $y = mx + k$. Let (x_1, y_1), (x_2, y_2) stand for the coordinates of P_1, P_2, respectively, and let $P(x_0, y_0)$ be the middle point of the chord. Then by (6), p. 9,

$$x_0 = \tfrac{1}{2}(x_1 + x_2), \qquad y_0 = \tfrac{1}{2}(y_1 + y_2). \qquad (i)$$

To express the values of x_1, x_2, y_1, y_2 in terms of the given constants, p, m, and k the equations $y^2 = 2px$ and $y = mx + k$ must be solved simultaneously for x and y. Eliminating x we have the quadratic

$$my^2 - 2py + 2pk = 0, \qquad (ii)$$

whose roots are y_1 and y_2, the ordinates of P_1 and P_2 respectively. Hence from (i), (ii) and II (c), p. viii,

$$y_0 = \tfrac{1}{2}(y_1 + y_2) = \frac{p}{m} \qquad (iii)$$

Since the value of y_0 in (iii) does not contain k, it follows that the middle points of *all chords having the slope m* are at the same distance p/m from the axis of the parabola. Hence there

is no necessity for finding the values of x, and the line

$$y = \frac{p}{m} \tag{15}$$

parallel to the axis of the curve is the diameter corresponding to this system of parallel chords.

To prove that every line $y = c$ is a diameter of the parabola $y^2 = 2px$, it is only necessary to write $c = p/m$, from which $m = p/c$, which will be the slope of the system of chords bisected by the line $y = c$, as appears from (15). Hence $y = c$ is a diameter of the parabola.

III. THEOREM. *Every diameter of an ellipse is a straight line passing through the center.*

Let $y = mx + k$ be the equation of the line MN, with given slope m, meeting the ellipse $x^2/a^2 + y^2/b^2 = 1$ at P_1, P_2, whose coordinates will be represented by (x_1, y_1), (x_2, y_2).

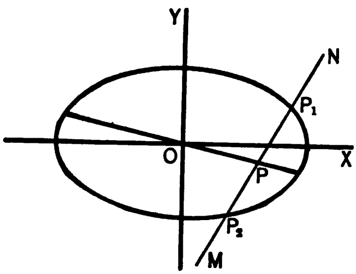


FIG. 64

Let $P(x_0, y_0)$ be the middle point of P_1P_2. Then as in the demonstration of II

$$x_0 = \tfrac{1}{2}(x_1 + x_2),$$
$$y_0 = \tfrac{1}{2}(y_1 + y_2).$$

To find the values of x_1, x_2, y_1, y_2 in terms of a, b, m, and k solve the equations of the ellipse and line simultaneously. Eliminating y

$$(a^2m^2 + b^2)x^2 + 2a^2mkx + a^2(k^2 - b^2) = 0,$$

hence as in II

$$x_0 = \tfrac{1}{2}(x_1 + x_2) = -\frac{a^2mk}{a^2m^2 + b^2}. \tag{i}$$

Substituting this value of x_0 for x in $y = mx + k$

$$y_0 = \frac{b^2k}{a^2m^2 + b^2}. \tag{ii}$$

Dividing (*ii*) by (*i*) k is eliminated, and therefore the result

$$\frac{y_0}{x_0} = -\frac{b^2}{a^2m}$$

is true for the middle point of *every* chord which has the slope m. Hence the equation of the locus of these middle points, that is the equation of the diameter bisecting the system of chords whose slope is m, is

$$y = -\frac{b^2}{a^2m}x. \tag{16}$$

This equation shows (1) that the diameter of an ellipse is a straight line, and (2) that it passes through the origin. As the origin is at the center of the ellipse the theorem is proved.

IV. THEOREM. *Every diameter of a hyperbola is a straight line passing through the center.*

This theorem is proved in the same manner as III, and the result is obtained simply by changing the sign of b^2 throughout. Hence

$$y = \frac{b^2}{a^2m}x \tag{17}$$

is the equation of the diameter of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ which bisects the system of parallel chords which have the slope m.

V. The points where the diameter of an ellipse or hyperbola meets the curve are called the **extremities** of the diameter.

70. Conjugate diameters of ellipse and hyperbola.

I. DEFINITION. *A pair of diameters of an ellipse or hyperbola, each of which bisects the system of chords parallel to the other, are called* ***conjugate diameters.***

II. THEOREM. *If m and m' are the slopes of two diameters of an ellipse, they will be conjugate diameters if*

$$mm' = -\frac{b^2}{a^2}. \tag{18}$$

Let m' be the slope of the diameter of the ellipse which bisects the system of parallel chords having the slope m, then from (16)

$$m' = -\frac{b^2}{a^2 m}, \qquad \text{or} \qquad mm' = -\frac{b^2}{a^2}.$$

III. THEOREM. *If m and m' are the slopes of two diameters of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$, they will be conjugate diameters if*

$$mm' = \frac{b^2}{a^2}. \tag{19}$$

The proof is left to the student.

EXERCISES. 1. Find for each of the conics below the equation of the diameter which bisects the system of parallel chords whose slope is the number given, and determine the point or points where the diameter meets the conic.

(a) $y^2 = 6x$, $m = \frac{1}{2}$; (c) $4x^2 + 9y^2 = 36$, $m = 1$;
(b) $y^2 = 4x$, $m = 1$; (d) $4x^2 - 9y^2 = 36$, $m = 2$.

Ans. (a) $(6, 6)$, (c) $(\frac{9}{13}\sqrt{13}, -\frac{4}{13}\sqrt{13})$, $(-\frac{9}{13}\sqrt{13}, \frac{4}{13}\sqrt{13})$.

2. Construct each of the following conics and mark on it the point whose coordinates are indicated. Find (*i*) the equation of the diameter to the point indicated, and (*ii*) the slope of the chords which the diameter bisects. Construct three chords of the system.

(a) $y^2 = 4x$, $(4, 4)$. Ans. (*i*) $y = 4$, (*ii*) $\frac{1}{2}$.
(b) $4x^2 + 9y^2 = 40$, $(1, -2)$. Ans. (*i*) $y = -2x$, (*ii*) $\frac{2}{9}$.
(c) $x^2 - 4y^2 = 9$, $(5, 2)$. (d) $x^2 - y^2 = 16$, $(5, -3)$.

3. In Exs. 1 (c) and 2 (b) find (*i*) the equation of the diameter conjugate to the one already found, (*ii*) the coordinates of the points where this conjugate diameter meets the curve.

4. For each hyperbola of Exs. 1 and 2 find (*i*) the equation of the diameter conjugate to the one already found, (*ii*) the coordinates of the points where this conjugate diameter intersects the conjugate hyperbola. Ans. to Ex. 2 (c), $y = \frac{5}{8}x$, $(4, \frac{5}{2})$, $(-4, -\frac{5}{2})$.

5. What is the diameter conjugate (a) to the major axis of an ellipse, (b) to the transverse axis of a hyperbola?

71. Pole and polar with respect to a conic.

In the following discussion the results will be worked out in detail only for the circle. From these it will be easy to infer the corresponding results for the conics in general, or to work them out independently.

I. DEFINITION. *The line joining the points of contact of two tangents to a circle (or conic) is called the* **chord of contact** *of the two tangents.*

II. THEOREM. *If two tangents to the circle $x^2 + y^2 = r^2$ meet at the point (x_0, y_0) the equation of the chord of contact is $x_0x + y_0y = r^2$.*

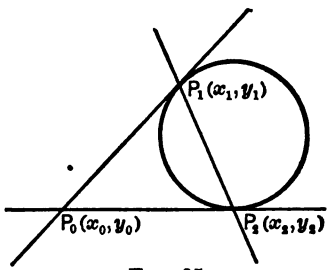


FIG. 65

The coordinate axes are omitted from the figure to save useless complication. Let the equation of the circle be $x^2 + y^2 = r^2$, and let the points of tangency of the tangents which meet at (x_0, y_0) be (x_1, y_1) and (x_2, y_2) Then the equations of these tangents will be

$$x_1x + y_1y = r^2, \qquad x_2x + y_2y = r^2. \tag{i}$$

Since both of these lines pass through (x_0, y_0) we have identically

$$x_1x_0 + y_1y_0 = r^2, \qquad x_2x_0 + y_2y_0 = r^2. \tag{ii}$$

Then it is clear that

$$x_0x + y_0y = r^2 \tag{20}$$

is the equation of P_1P_2, because, being of the first degree it is the equation of a straight line, and if the coordinates of P_1 and P_2 be substituted for x and y it is evident from (*ii*) that it is satisfied. Hence the line (20) passes through P_1 and P_2.

III. THEOREM. *If through a fixed point $P_0(x_0, y_0)$ in the plane of a circle (or conic) a straight line be drawn meeting the*

circle (or conic) in the points A and B, and if tangents be drawn to the circle (or conic) at the points A and B, the locus of the intersection of these tangents will be a straight line.

Let the equation of the circle be $x^2 + y^2 = r^2$, as in II. The demonstration applies to both figures. Let the two tangents

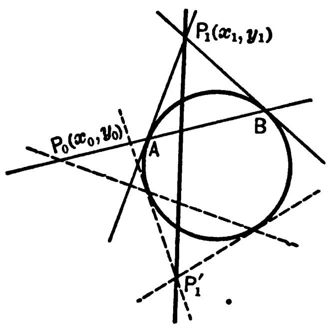


FIG. 66(a)

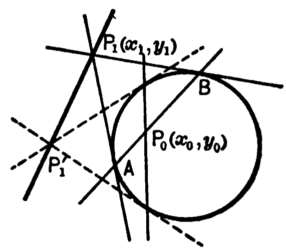


FIG. 66(b)

meet at $P_1(x_1, y_1)$ then AB is the chord of contact of P_1, and hence by (20) the equation of AB is

$$x_1x + y_1y = r^2. \tag{i}$$

Since the line (*i*) passes through (x_0, y_0)

$$x_1x_0 + y_1y_0 = r^2. \tag{ii}$$

Now let AB move, but always in such a way that it passes through P_0, then it is clear from (*ii*) that

$$x_0x + y_0y = r^2 \tag{21}$$

is the equation of a straight line which is satisfied by the coordinates of P_1 in all of its positions, hence this line (21) is the locus of P_1.

Since the locus of P_1 is a straight line, it may be constructed for any given position of P_0, by drawing two secants through

P_0, constructing the two pairs of tangents at the points where the secants intersect the curve, and connecting the two points where the two pairs of tangents meet. In Figs. 66 (a) and (b) the construction is completed with dotted lines, and P_1P_1' is the required locus.

IV. DEFINITION. *The line whose equation has been derived in* (21) *is called the* **polar** *of the point* P_0 *with respect to the curve, and the point* P_0 *is called the* **pole** *of the line.*

72. Discussion and extension of the results of Art. 71. Equations (20) and (21) have the same form as that of the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_0, y_0). Hence the tangent to a circle and the chord of contact of a point are special cases of the more general relation of pole and polar. In all these cases the *form* of the equation is the same, the only difference being in the location with respect to the circle of the fixed point (x_0, y_0) whose coordinates appear in the equation.

If in Art. 71 the equation of a conic had been used instead of that of a circle the nature of the results would have been the same.

EXERCISES. 1. Write the equations of the polars of the points (6, 4), (4, 4), (2, 4) with respect to each of the conics given below, and construct separate figures for each curve with its poles and polars.

(a) $x^2 + y^2 = 32$, (b) $3x^2 + y^2 = 64$, (c) $y^2 = 4x$,
(d) $4x^2 - y^2 = 48$, (e) $x^2 = 4y$, (f) $xy = 16$.

Ans. (a) $3x + 2y = 16,\ x + y = 8,\ x + 2y = 16$.
(c) $2y = x + 6,\ 2y = x + 4,\ 2y = x + 2$.

2. Show that for any conic the polar of a focus is the corresponding directrix.

EXERCISES ON CHAPTER VI

Normal Exercises

1. For each conic below find the equation of the tangent or tangents having the slope indicated. Check the results graphically.

(a) $81x^2 + 175y^2 = 63,\quad m = -\frac{3}{5}$; ($b$) $y^2 = 6x,\quad m = 2$;

(c) $16x^2 - 25y^2 = 16$, $m = 1$; (d) $9x^2 + 9y^2 = 25$, $m = \frac{3}{4}$.

Ans. (a) $3x + 5y \pm 4 = 0$, (b) $8x - 4y + 3 = 0$.

2. Determine the equations of the tangent and normal to each conic below at the point indicated. Check the results graphically.

(a) $x^2 + 4y^2 + 2x + 8y - 20 = 0$, $(2, -3)$;
(b) $x^2 - y^2 - 4x + 6y + 7 = 0$, $(4, -1)$;
(c) $16x^2 + 16y^2 - 25 = 0$, $(1, -\frac{3}{4})$; (d) $y^2 + 8x = 0$, $(-2, -4)$;
(e) $x^2 + 9y^2 = 40$, $(-2, 2)$; (f) $x^2 - y^2 + 16 = 0$, $(-3, 5)$;
(g) $4x^2 - 9y^2 = 7$, $(2, -1)$; (h) $xy = 6$, $(2, 3)$.

Ans. (a) Tan. $3x - 8y = 30$, Nor. $8x + 3y = 7$.
(b) Tan. $x + 2y - 2 = 0$, Nor. $2x - y - 9 = 0$.

3. For each conic below find the equation of the diameter which bisects the system of parallel chords having the given slope. Construct the figures.

(a) $y^2 = 6x$, $m = -3$; (b) $4x^2 + y^2 = 36$, $m = \frac{1}{2}$;
(c) $4x^2 - y^2 = 36$, $m = \frac{1}{2}$.

4. For each conic below find the equation of the diameter to the point indicated, and the slope of the chords which it bisects. Construct the figures.

(a) $2x^2 + 3y^2 = 50$, $(-1, -4)$; (b) $y^2 = 6x$, $(2, 2\sqrt{3})$;
(c) $4x^2 - y^2 = 32$, $(-3, 2)$. Ans. (a) $y = 4x$, $-\frac{1}{6}$.

5. For each ellipse and hyperbola in exercises 3 and 4 find the equation of the diameter conjugate to the one indicated.

6. For each conic below find the equation of the chord of contact of the point indicated. Construct the figures.

(a) $x^2 + y^2 = 36$, $(2, 6)$; (b) $4x^2 + 9y^2 = 100$, $(3, 4)$;
(c) $y^2 + 4x = 0$, $(2, -3)$; (d) $x^2 - 4y^2 = 16$, $(1, 3)$;
(e) $xy = 5$, $(2, 2)$; (f) $4x^2 - y^2 + 16 = 0$, $(1, 3)$.

7. For each of the following conics determine the equation of the polar of the point indicated. Construct the figures.

(a) $x^2 + y^2 = 49$, $(-1, 1)$; (b) $4x^2 + 9y^2 = 36$, $(2, 1)$;
(c) $4x^2 - 9y^2 = 36$, $(4, 1)$; (d) $x^2 - y^2 + 16 = 0$, $(8, 1)$
(e) $y^2 = 8x$, $(2, -3)$; (f) $xy = 4$, $(-4, -2)$.

General Exercises

8. Find the equations of the tangents to $4x^2 + 16y^2 = 25$ which are parallel to $4x - 5y + 7 = 0$.

9. Find the equations of the tangents to $x^2 - 4y^2 = 9$ which are perpendicular to $2x - 4y - 3 = 0$.

10. Using equation (7) determine the equations of the two tangents to $y^2 = 4x$ which pass through the point $(-4, 3)$.

Ans. $x + y + 1 = 0,\ x - 4y + 16 = 0$.

11. Derive the equations of the lines with slope $\sqrt{2}$, which are tangent to $15x^2 - 8y^2 + 30x + 32y - 137 = 0$.

Ans. $y = \sqrt{2}x + 3 + \sqrt{2},\ y = \sqrt{2}x + 1 + \sqrt{2}$.

12. Find the equation of the normal to $xy = 4$ which has the slope $\frac{1}{4}$.

Ans. $x - 4y \pm 15 = 0$.

13. Find the equation of the normal to $y^2 = 4x$ which is perpendicular to $3x - 4y = 0$. Ans. $36x + 27y = 136$.

14. Find the equation of the normal to $8x^2 + 9y^2 = 72$ whose slope is 1. Ans. $\sqrt{17}x - \sqrt{17}y \pm 1 = 0$.

15. Prove that the line joining the center of a circle to any point is perpendicular to the polar of the point with respect to the circle.

16. Two perpendicular lines are tangent to $y^2 = 4x$, one of them touching the curve at $(1, -2)$. Find the equations of these tangents, and show that they meet on the directrix.

17. Two perpendicular lines are tangent to $x^2 + 4y^2 = 20$, one of which touches the curve at $(4, 1)$. Find the equations of these tangents and show that they intersect on the circle $x^2 + y^2 = r^2$, where r^2 is equal to the sum of the squares of the semi-axes of the given ellipse.

18. Find the equations of the normals to the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ which pass through the center. Use equation (13), p. 115.

19. Find the coordinates of those points of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ the normals at which pass through a focus.

20. If in an ellipse the normal at one extremity of the latus rectum passes through the opposite end of the minor axis, what is the eccentricity of the ellipse? Ans. $\sqrt{\frac{1}{2}\sqrt{5} - \frac{1}{2}}$.

21. Given the coordinates (x_1, y_1) of one extremity of a diameter of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, determine the coordinates of the extremities of the conjugate diameter.

Ans. $(-(a/b)y_1,\ (b/a)x_1)$ and $((a/b)y_1,\ -(b/a)x_1)$.

22. Using the results of exercise 21 show that if a' and b' are the lengths of any two conjugate semi-diameters of an ellipse, then

$$a'^2 + b'^2 = a^2 + b^2.$$

23. Given the coordinates (x_1, y_1) of one extremity of a diameter of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$, determine the coordinates of the extremities of the conjugate diameter.

Ans. $((a/b)y_1, (b/a)x_1)$ and $(-(a/b)y_1, -(b/a)x_1)$.

24. Using the results of exercise 23 show that if a' and b' are the lengths of any two conjugate semi-diameters of a hyperbola, then $a'^2 - b'^2 = a^2 - b^2$.

25. Prove that the tangent at any point of (*i*) a parabola, (*ii*) an ellipse, (*iii*) a hyperbola is parallel to the chords bisected by the diameter drawn to that point.

26. For each of the conics below determine the equation of the chord whose middle point is the point indicated

(*a*) $y^2 = 4x$, $(2, 1)$. (*b*) $x^2 + 4y^2 = 16$, $(1, 1)$.
(*c*) $y^2 = 6x$, $(3, -3)$. (*d*) $x^2 - 4y^2 = 16$, $(1, -1)$.

Ans. (*a*) $2x - y - 3 = 0$, (*b*) $x + 4y - 5 = 0$.

27. Find the equations of the two conjugate diameters of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ whose lengths are equal. Ans. $y = \pm (b/a)x$.

28. Find for the following conics the equations of the two tangents to the curve from the point indicated in each case.

(*a*) $y^2 = 4x$, $(2, -3)$. (*b*) $8x^2 + 21y^2 = 221$, $(-\frac{17}{3}, -\frac{17}{3})$.
(*c*) $xy = 2$, $(2, -3)$. (*d*) $9x^2 - 25y^2 = 225$, $(5, 2)$.

Ans. (*a*) $x + y + 1 = 0$, $x + 2y + 4 = 0$.
(*b*) $16x - 63y - 221 = 0$, $40x - 21y + 221 = 0$.

29. Using the method of Art. 67, derive the equations of the tangent and normal to each of the following curves at the point indicated:

(*a*) $y = x^3 + 2x$, $(1, 3)$. (*b*) $y = 2x^3 + 3$, $(-2, -13)$.
(*c*) $y = x^4 - 1$, $(2, 15)$. (*d*) $y^2 = x^3$, $(1, 1)$.

Ans. (*a*) Tan. $5x - y - 2 = 0$, Nor. $x + 5y - 16 = 0$.
(*b*) Tan. $24x - y + 35 = 0$, Nor. $x + 24y + 314 = 0$.

CHAPTER VII

APPLICATIONS OF ANALYTIC GEOMETRY

73. Remarks and definitions.—Analytic geometry is, in the hands of the mathematician, a powerful instrument for proving theorems relating to, and discussing the properties of, curves and loci whose equations are known. This chapter will be devoted to an explanation of some of the methods used. They will be illustrated by the discussion of properties of the conics.

DEFINITIONS. I. *The distance on the X-axis from the point where a tangent to a curve cuts this axis, to the foot of the ordinate at the point of tangency, is called the* **sub-tangent** *of the point.*

Thus TM is the subtangent of the point P.

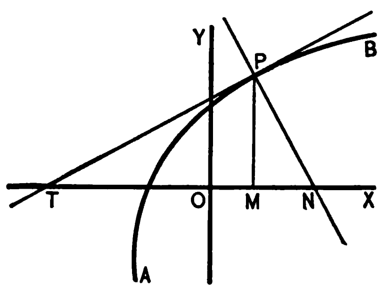


FIG. 67

II. *The distance on the X-axis from the foot of the ordinate at any point on a curve, to the intersection with the X-axis of the normal to the curve at that point, is called the* **sub-normal** *of the point.*

Thus MN is the subnormal of the point P.

In other words the subtangent and subnormal of a point on a curve are the projections on the X-axis of the portions of the tangent and normal respectively which lie between the point of tangency and the intersection with the X-axis.

74. Subtangent and subnormal of a point on the parabola.

I. *The length of the subtangent of* $P(x_1, y_1)$ *on the parabola* $y^2 = 2px$ *is* $2x_1$.

The equation of the tangent TP to the parabola $y^2 = 2px$, at the point $P(x_1, y_1)$ is (7), p. 113,

$$yy_1 = p(x + x_1).$$

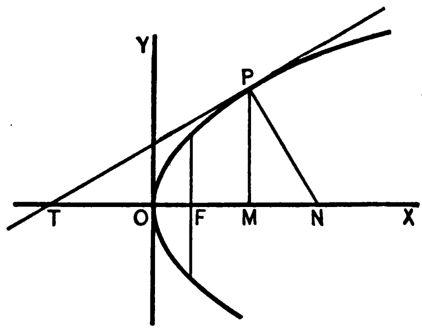


FIG. 68

Making $y = 0$, we have $x = -x_1$, which is therefore the intercept OT of TP on $X'X$. But if $OT = -x_1$, then $TO = x_1$, and hence the length of the subtangent TM is

$$TM = TO + OM = x_1 + x_1 = 2x_1. \tag{1}$$

II. *The length of the subnormal of any point on the parabola* $y^2 = 2px$ *is constant and equal to* p, *the semi-latus rectum.*

The equation of the normal PN to $y^2 = 2px$ at $P(x_1, y_1)$ is (12), p. 115,

$$y - y_1 = -\frac{y_1}{p}(x - x_1).$$

Make $y = 0$, then $x = x_1 + p$, which is the length of ON, the x-intercept of PN. Hence the length of the subnormal MN is

$$MN = MO + ON = -x_1 + (x_1 + p) = p. \tag{2}$$

Let the student deduce from I and II two methods for constructing the tangent to a parabola at a given point.

75. Deductions from Art. 74, I.

I. *The tangent at any point of a parabola makes equal angles with the axis of the curve and the focal radius to the point of tangency.*

The notation is the same as in Art. 74, and also the figure, with the addition of the focal radius FP, and the directrix, which intersects the axis of the parabola at D. See Fig. 69.

From the definition of the parabola

$$FP = DM = DO + OM = \tfrac{1}{2}p + x_1,$$

and from Art. 74, I

$$TF = TO + OF = x_1 + \tfrac{1}{2}p,$$

hence

$$FP = TF.$$

The triangle TFP, therefore, is isosceles, and

$$\textbf{angle } FTP = \textbf{angle } FPT. \tag{3}$$

Hence, also:

II. *The normal PN at any point on the parabola bisects the angle between the focal radius FP, and the diameter PL through the point.*

76. Additional properties of the parabola.

I. *The perpendicular from the focus of a parabola to any tangent meets the latter on the tangent at the vertex.*

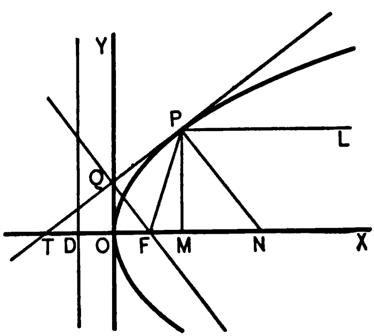


FIG. 69

Let $y^2 = 2px$ be the parabola, and draw FQ perpendicular to the tangent TP, Fig. 69. To prove the theorem either form of the equation of the tangent, (4), p. 110, or (7),

p. 113, may be used. Let us choose the equation in terms of the slope

$$y = mx + \frac{p}{2m}. \qquad (i)$$

Hence FQ, the line through $F(\frac{1}{2}p, 0)$, perpendicular to PT, has the equation

$$y = -\frac{1}{m}(x - \tfrac{1}{2}p). \qquad (ii)$$

Eliminating y from (i) and (ii), and solving for x, we have after reduction,

$$x = 0.$$

Therefore, the intersection of PT and FQ lies on the Y-axis which is the tangent at the vertex.

II. *The perpendicular from the focus of a parabola to any tangent meets the diameter through the point of tangency on the directrix.*

If (x_1, y_1) be the point of tangency the equation of the tangent is $yy_1 = p(x + x_1)$ and that of the diameter is $y = y_1$. The details of the proof are left to the student.

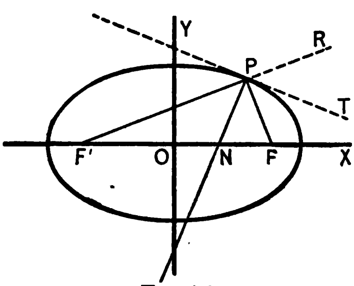


FIG. 70

77. Theorems on the ellipse.

I. *The normal at any point of the ellipse bisects the angle between the focal radii drawn to that point.*

Let $(x^2/a^2) + (y^2/b^2) = 1$ be the ellipse, then the equation of the normal at the point $P(x_1, y_1)$ is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1). \qquad (i)$$

It is proved in elementary geometry that if a line PN drawn from the vertex P of a triangle FPF' divides the opposite side $F'F$ in segments $F'N$, NF which are proportional to the adjacent sides $F'P$, FP, then this line bisects the angle $F'PF$. Hence, PN being the normal to the ellipse at P, if the proportion

$$\frac{F'N}{NF} = \frac{F'P}{FP} \tag{ii}$$

is true, PN bisects the angle $F'PF$.

To show this let $y = 0$ in (i), then

$$x = ON = \frac{a^2 - b^2}{a^2}x_1 = e^2x_1, \text{ by (11), p. 87,}$$

and, since $F'O = OF = ae$,

$$F'N = F'O + ON = ae + e^2x_1 = e(a + ex_1),$$

$$NF = OF - ON = ae - e^2x_1 = e(a - ex_1),$$

$$\therefore \qquad \frac{F'N}{NF} = \frac{a + ex_1}{a - ex_1}. \tag{iii}$$

Also by (ii) and (iii), Art. 54,

$$\frac{F'P}{FP} = \frac{a + ex_1}{a - ex_1}. \tag{iv}$$

Hence from (iii) and (iv) it follows that (ii) is true, and therefore the theorem is proved.

Since PN bisects $F'PF$ it follows also that the tangent PT bisects the angle FPR.

This property connecting the tangent and normal at a point of the ellipse with the focal radii drawn to that point affords a convenient method for constructing the tangent and normal to an ellipse at a given point.

II. DEFINITION. *The circles concentric with a given ellipse, and having for diameters the major and minor axes of the ellipse,*

are called respectively the **major auxiliary circle** *and the* **minor auxiliary circle** *of the ellipse.*

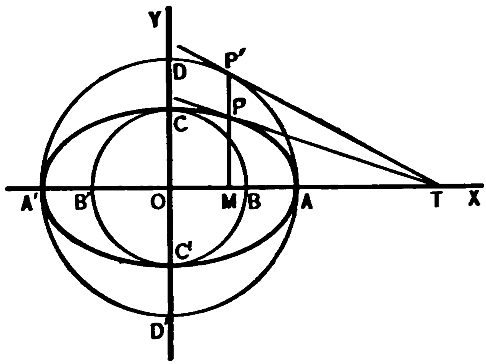


FIG. 71

Thus the circle $ADA'D'$ is the major auxiliary circle of the ellipse $ACA'C'$, and the circle $BCB'C'$ is the minor auxiliary circle.

III. DEFINITION. *A pair of points P, P', Fig. 71, on an ellipse and on its major auxiliary circle, determined by a perpendicular to the major axis, are called* **corresponding points.**

IV. THEOREM. *The ordinates to a pair of corresponding points on an ellipse and its major auxiliary circle are to each other respectively as b is to a.*

Taking the coordinate axes as indicated in Fig. 71 the equation of the ellipse $ACA'C'$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (i)$$

and that of the major auxiliary circle $ADA'D'$ is

$$x^2 + y^2 = a^2. \qquad (ii)$$

The corresponding points P, P' have the same abscissa $OM = x$.

Designate MP by y, and MP' by y'. Then from (i)

$$MP = y = \frac{b}{a}\sqrt{a^2 - x^2}, \tag{iii}$$

and from (ii)

$$MP' = y' = \sqrt{a^2 - x^2}. \tag{iv}$$

Dividing (iii) by (iv)

$$\frac{y}{y'} = \frac{b}{a}. \tag{4}$$

78. The eccentric angle of a point on the ellipse.

I. DEFINITION. *The* ***eccentric angle*** *of a point on the ellipse is the angle between the major axis and the line connecting the center with the corresponding point on the major auxiliary circle.*

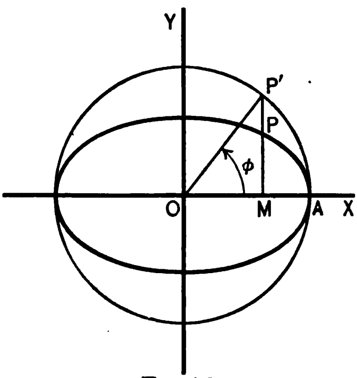


FIG. 72

Thus, the angle $AOP' = \phi$ is the eccentric angle of the point P on the ellipse.

II. THEOREM. *The coordinates of a point on the ellipse in terms of the eccentric angle are*

$$\boldsymbol{x = a \cos \phi, \qquad y = b \sin \phi.} \tag{5}$$

From the right triangle MOP'

$$OM = OP' \cos \phi, \qquad \text{or} \qquad x = a \cos \phi,$$

since $OP' = OA = a$.

Also, $MP' = OP' \sin \phi$, or, letting $MP' = y'$, $y' = a \sin \phi$. Hence from (4),

$$MP = y = b \sin \phi.$$

When the coordinates of a point on a locus are thus expressed

in terms of a third variable, this variable is called a **parameter,** and the equations (5) are called the **parametric equations** of the locus.

79. Additional methods of constructing an ellipse.

I. *Construction of an ellipse by two pins and a string.* The major and minor axes, AA', BB', are supposed to be given. Locate the foci F, F' by making $BF = BF' = CA$. Fix two pins firmly at F and F', and with a piece of string make a loop over the pins of such length that, when stretched taut with a pencil in the loop, the point of the pencil will be at B. Move the pencil, keeping the string taut. Then the pencil will describe the ellipse $BA'B'A$, because if P be any position of the pencil the sum of the two distances

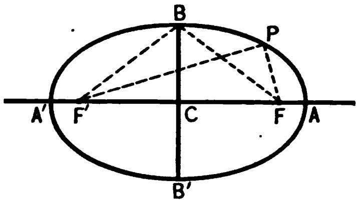


FIG. 73

$$F'P + FP = F'B + FB = A'A.$$

See Art. 54.

II. *Another method of construction.* Upon the straight edge AB of a piece of paper or cardboard mark off PR equal to the semi-major axis of the required ellipse, and PQ equal to the semi-minor axis. Draw $X'X$ and $Y'Y$ at right angles, and place the edge of the marked paper so that R will fall on $Y'Y$, and Q on $X'X$. Then P will be a point on the ellipse which has the given semi-axes and center at O.

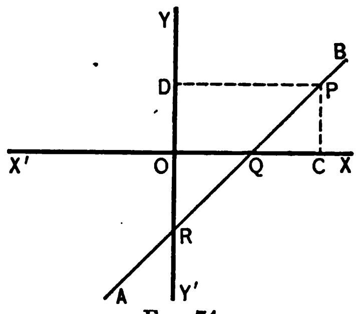


FIG. 74

To prove that this construction will give points on the ellipse

with semi-axes $RP = a$, $QP = b$, take $X'X$, $Y'Y$ as axes of coordinates, and let angle $XQB = \theta$. Then the coordinates of P are

$$x = DP = RP \cos\theta = a \cos\theta,$$

$$y = CP = QP \sin\theta = b \sin\theta.$$

Since these coordinates have the form (5), p. 133, the locus of the point $P(x, y)$ is an ellipse.

80. Theorems on the hyperbola.

I. *The tangent to a hyperbola at any point bisects the angle between the focal radii drawn to that point.*

The proof, which is similar to that of Art. 77, I, is left to the student.

II. *The portion of any tangent to the hyperbola, which is included between the asymptotes, is bisected at the point of tangency.*

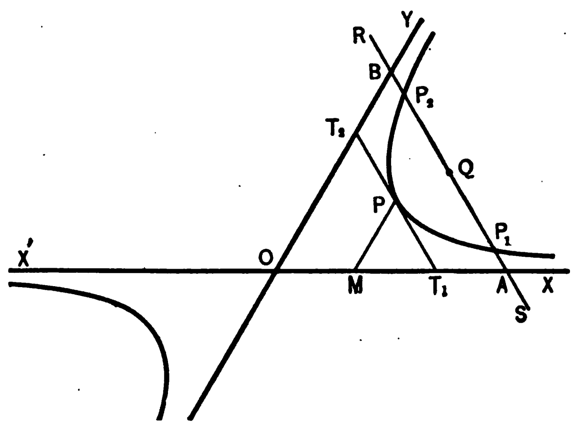


Fig. 75

Taking the asymptotes as axes of coordinates, the equation of the hyperbola will be

$$xy = c, \qquad (i)$$

and that of the tangent T_1T_2 at $P(x', y')$, see (10) p. 115

$$y'x + x'y = 2c. \qquad (ii)$$

From (ii) making $y = 0$, the x-intercept OT_1 of the tangent is

$$x = \frac{2c}{y'} = \frac{2x'y'}{y'} = 2x'.$$

That is

$$OT_1 = 2x' = 2OM.$$

Hence MP which is parallel to OT_2 bisects T_1T_2, since it bisects OT_1, which proves the theorem.

81. Loci by elimination of parameters.

I. *Perpendicular tangents to the parabola meet on the directrix.*

Let $y^2 = 2px$ be any parabola, then

$$y = mx + \frac{p}{2m}, \qquad y = -\frac{1}{m}x - \frac{mp}{2}, \qquad (i)$$

will be the equations of any two perpendicular tangents. Equations (i) may be written in the form

$$my - m^2x = \tfrac{1}{2}p, \qquad my + x = -\tfrac{1}{2}m^2p,$$

from which, subtracting the first from the second,

$$(m^2 + 1)x = -\tfrac{1}{2}p(m^2 + 1),$$

or

$$x = -\tfrac{1}{2}p. \qquad (ii)$$

This result gives the value of the abscissa of the point of intersection of the two perpendicular tangents (i). Since it does not contain m, this same value will result for all pairs of perpendicular tangents. Hence $x = -\frac{1}{2}p$ is the equation of the locus of the point of intersection of all pairs of perpendicular tangents. It is the equation of the directrix, and therefore the theorem is proved.

II. VARIABLE PARAMETERS. When in an equation a quantity, other than the coordinates x and y, is assumed to be variable, this quantity is called a **variable parameter.** Thus in equations (i) m is a variable parameter if we take these equations to represent not a particular pair of perpendicular tangents, but all possible pairs of perpendicular tangents as m takes in succession all possible values.

When the equations of two loci are given containing the same variable parameter, and this parameter is eliminated between them, the resulting equation in x and y is that of the locus of the intersection of the two given loci, because the new equation thus derived expresses a relation between x and y which holds true simultaneously for the coordinates (x, y) of all the points of intersection. Thus in I above by eliminating the variable parameter m between the two equations (i) there results an equation (ii) which is satisfied by the coordinates of the points of intersection of all the pairs of tangents represented by equations (i), and is therefore the equation of the locus of their intersection. This process is further illustrated in III and IV below.

NOTE. The fact that y is eliminated simultaneously with m in I is purely incidental to this particular problem.

In cases where more than one variable parameter occur the same general principles apply. One or more additional equations between the parameters are then necessary, however, in order to effect the elimination of the parameters.

III. *The major auxiliary circle of an ellipse is the locus of the intersection of any tangent to the ellipse with the perpendicular upon it from the focus.*

The equation of the tangent to the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ in terms of the slope is, by (1), p. 109,

$$y = mx \pm \sqrt{a^2m^2 + b^2}, \tag{i}$$

The perpendicular to (*i*) through the focus $(ae, 0)$ is

$$y = -\frac{1}{m}(x - ae). \qquad (ii)$$

To obtain the locus of the intersection of these two lines the variable parameter m must be eliminated from (*i*) and (*ii*). From (*i*)

$$y - mx = \pm\sqrt{a^2m^2 + b^2}, \qquad (iii)$$

and from (*ii*)

$$my + x = ae = \sqrt{a^2 - b^2}. \qquad (iv)$$

Squaring (*iii*) and (*iv*)

$$y^2 - 2mxy + m^2x^2 = a^2m^2 + b^2, \qquad (v)$$

$$m^2y^2 + 2mxy + x^2 = a^2 - b^2. \qquad (vi)$$

Adding (*v*) and (*vi*)

$$(1 + m^2)y^2 + (1 + m^2)x^2 = a^2(1 + m^2)$$

or

$$x^2 + y^2 = a^2,$$

the equation of the required locus.

IV. *Perpendicular tangents to the ellipse meet on the circle* $x^2 + y^2 = a^2 + b^2$.

The equations of a pair of perpendicular tangents to the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ are

$$y = mx \pm \sqrt{a^2m^2 + b^2}, \qquad (i)$$

and

$$y = -\frac{1}{m}x \pm \sqrt{\frac{a^2}{m^2} + b^2}, \qquad (ii)$$

from which m must be eliminated. From (*i*)

$$y - mx = \pm\sqrt{a^2m^2 + b^2}, \qquad (iii)$$

and from (*ii*)

$$my + x = \pm\sqrt{a^2 + m^2b^2}. \qquad (iv)$$

Squaring and adding (*iii*) and (*iv*)

$$(1 + m^2)y^2 + (1 + m^2)x^2 = (1 + m^2)a^2 + (1 + m^2)b^2,$$

or

$$x^2 + y^2 = a^2 + b^2.$$

V. DEFINITION. *The circle $x^2 + y^2 = a^2 + b^2$ is called the* ***director circle*** *of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$.*

EXERCISES ON CHAPTER VII

1. Draw FP from the focus F to any point P on a parabola, and FL perpendicular to FP at F. Show that FL and the tangent to the parabola at P meet on the directrix.

2. From any point P of a parabola let fall PD, perpendicular to the axis of the curve at D. Let K be another point on the parabola at a distance from the axis $= \frac{1}{2}DP$. Draw DK, and produce it to intersect the tangent at the vertex V, at R. Prove $VR = \frac{2}{3}DP$.

3. The tangent at a point P on a parabola meets the directrix at R, and the perpendicular to the axis through the focus F, at Q. Prove $FR = FQ$.

4. Prove that in any parabola the tangent at one extremity of the latus rectum is parallel to the normal at the other extremity.

5. Let the tangents at any two points P and Q on a parabola meet at R, and let S be the middle point of the chord PQ. Draw RS. Prove (*i*) that T, the middle point of RS, is on the parabola, (*ii*) that RS is parallel to the axis of the parabola, (*iii*) that PQ is parallel to the tangent at T.

6. Prove that the line from the focus of a parabola to the intersection of any two tangents bisects the angle between the lines from the focus to the points of tangency.

7. Prove (*i*) that the tangents at the extremity of any focal chord of a parabola meet on the directrix, (*ii*) that these tangents are perpendicular to each other, (*iii*) that the chord is perpendicular to the line joining the focus and the intersection of the tangents.

8. Prove that the circle constructed on any focal chord of a parabola as diameter is tangent to the directrix.

9. Prove that if a circle be constructed with any focal radius FP of a

parabola as diameter, it will be tangent to the tangent at the vertex of the parabola.

10. Given a parabola with its axis and focus, describe six different methods of constructing geometrically a tangent to the parabola at any point on the curve.

11. Given a parabola, describe a method for constructing geometrically its axis, focus and directrix.

12. Prove that the perpendicular from the focus of an ellipse or hyperbola to any tangent, meets the line passing through the center and the point of tangency, on the corresponding directrix.

13. Prove that tangents to an ellipse and to its major auxiliary circle at corresponding points intersect on the major axis produced.

14. Prove that any two tangents to an ellipse or hyperbola intersect on the diameter which bisects the chord joining the points of tangency.

15. Prove that if a straight line intersects a hyperbola, the segments on this line included between the curve and its asymptotes, are equal.

16. Tangents are drawn to an ellipse or hyperbola from any point on one of the directrices. Prove that the line drawn through the points of tangency passes through the corresponding focus.

17. A perpendicular is erected to any focal chord of an ellipse or hyperbola at the focus. Prove that this perpendicular meets the tangent at either extremity of the chord, on the corresponding directrix.

18. Prove that in any ellipse or hyperbola the product of the perpendiculars from the foci upon any tangent is equal to the square of the semi-conjugate axis.

19. Perpendiculars are erected to the transverse axis of any ellipse or hyperbola at the focus F, and any other point M, meeting the curve in R and P respectively. Draw the tangent at R, and let it meet MP (produced) in K. Show that $MK = FP$.

20. If the vertex of a hyperbola is twice as far from the center as from the focus, find the slopes of the asymptotes.

21. Let the tangent at one vertex of a hyperbola cut the conjugate hyperbola at P and P'. Prove that the tangents at P and P' pass through the other vertex.

22. Prove that an ellipse and hyperbola having the same foci cut each other at right angles.

23. Let P be any point on a hyperbola, V the vertex, and C the

center. The tangent at P meets CV at M, and the tangent at V meets CP at S. Prove that MS and VP are parallel.

24. Prove that in an equilateral hyperbola any two perpendicular diameters are equal.

25. Prove that in any hyperbola all triangles formed by a tangent and the asymptotes are equal in area.

26. Prove that in any hyperbola, (*i*) the distance from a focus to an asymptote equals the semi-conjugate axis, (*ii*) the distance from the center to the foot of the perpendicular from a focus to an asymptote equals the semi-transverse axis.

27. Prove that in any ellipse or hyperbola the polar of any point of a diameter is parallel to the conjugate diameter.

28. Given an ellipse, with its axes, foci and directrices, describe three methods for constructing a tangent to the ellipse at any given point.

29. Given a hyperbola, with its axes, foci and directrices, describe three methods for constructing a tangent to the hyperbola at any given point.

30. Given an ellipse (the curve only) show how to determine geometrically, its center, axes, foci and directrices.

CHAPTER VIII

THE GENERAL EQUATION OF THE SECOND DEGREE

82. Introductory note.—From equations (28)–(32), pp. 99–100, it can be seen that the equation of every conic whose principal axis is parallel to one of the coordinate axes can be expressed in the form

$$Ax^2 + By^2 + Gx + Fy + C = 0, \tag{1}$$

that is, with no term in xy; and, conversely, from the discussion of Art. 64 it follows that every equation of the form (1) represents either a conic, a pair of straight lines, or an imaginary locus. To avoid unnecessary repetition the locus corresponding to equation (1) will always be referred to as a conic. This term will therefore include pairs of straight lines, which will be called **degenerate conics,*** and imaginary loci, which will be referred to as **imaginary conics,** since their equations are of the same type as those of real conics.

Conics which are not degenerate or imaginary are called **proper conics.**

83. Theorem.—*The general or complete equation of the second degree in two variables*

$$Ax^2 + Hxy + By^2 + Gx + Fy + C = 0 \tag{2}$$

always represents a conic.

It should be observed that the coefficients A, H, B in equation (2) cannot all be zero, as the equation would then cease to be one of the second degree; and also that if $H = 0$ no proof is necessary, as the equation is then in the form (1). It is assumed therefore that $H \neq 0$.

* Sometimes also called composite conics.

Apply to (2) the formulas (26), p. 96, for turning the axes through the angle θ, without changing the origin,

$$x = x' \cos\theta - y' \sin\theta, \qquad y = x' \sin\theta + y' \cos\theta.$$

Substituting in (2) we have, after reduction

$$\left.\begin{aligned} &x'^2(A \cos^2\theta + H \sin\theta \cos\theta + B \sin^2\theta) \\ &\quad + x'y'[H(\cos^2\theta - \sin^2\theta) + 2(B - A) \sin\theta \cos\theta] \\ &\quad + y'^2(A \sin^2\theta - H \sin\theta \cos\theta + B \cos^2\theta) \\ &\quad + x'(G \cos\theta + F \sin\theta) + y'(F \cos\theta - G \sin\theta) + C = 0. \end{aligned}\right\} \quad (3)$$

The term in $x'y'$ will disappear from this equation if θ has a value which will make the coefficient of $x'y'$ equal to zero. Equating this coefficient to zero, and solving for θ, we have

$$H(\cos^2\theta - \sin^2\theta) + 2(B - A) \sin\theta \cos\theta = 0,$$

or

$$H \cos 2\theta + (B - A) \sin 2\theta = 0,$$

from which

$$\tan 2\theta = \frac{H}{A - B}. \qquad (4)$$

This always gives a real value of 2θ, and hence, by turning the axes through the angle θ thus determined, equation (2) can always be reduced to the form (1), which proves the theorem.

Exercises. Determine in each of the following cases the value of the angle θ through which the axes must be turned to reduce the equation to a form without the term in xy.

1. $2x^2 - 4xy + 5y^2 + 6x - 3y - 1 = 0.$ $\qquad \theta = \frac{1}{2} \tan^{-1} \frac{4}{3}.$
2. $3x^2 + 2xy - y^2 + 5x + 3y + 5 = 0.$ $\qquad \theta = \frac{1}{2} \tan^{-1} \frac{1}{2}.$
3. $x^2 + 3xy + y^2 - 6x + 2y + 7 = 0.$ $\qquad \theta = \frac{1}{2} \tan^{-1} \infty = 45°.$
4. $4x^2 - 4xy + y^2 + 2x - 6y - 10 = 0.$ $\qquad \theta = \frac{1}{2} \tan^{-1} (-\frac{4}{3}).$
5. $6x^2 - 5xy - y^2 + 4x + 10y - 7 = 0.$ $\qquad \theta = \frac{1}{2} \tan^{-1} (-\frac{5}{7}).$
6. $x^2 + 6xy + 9y^2 - 2x - 6y - 10 = 0.$ $\qquad \theta = \frac{1}{2} \tan^{-1} (-\frac{3}{4}).$

Note. No attempt is made at this point to go further and determine the complete form of the new equation, as it would be tedious and difficult to work out the values of the new coefficients by means of

equation (3). A better method for reaching this result will be given below. See Art. 86, II.

84. Central and non-central conics.

I. A conic is said to have a center when there is a single point in its plane which bisects all chords of the conic which pass through this point. Such a point is a center of symmetry of the curve, and is called its **center.**

II. *A conic which has a center is called a* **central conic.** Such are the ellipse and hyperbola.

III. *A conic which has no center is called a* **non-central conic.** Such is the parabola.

IV. THEOREM. *An equation of the second degree which contains no terms of the first degree in x and y, is the equation of a conic with center at the origin.*

If equation (2) contains no terms of the first degree in x and y. it becomes

$$Ax^2 + Hxy + By^2 + C = 0. \tag{i}$$

If this equation is satisfied by the coordinates (x_1, y_1), it is also satisfied by $(-x_1, -y_1)$. That is if $P(x_1, y_1)$ is on the conic, so also is $Q(-x_1, -y_1)$, and every chord PQ of the conic is bisected at O. Hence O is the center of the conic.

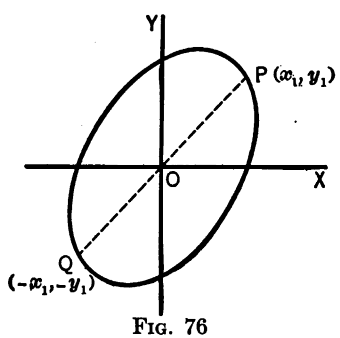


FIG. 76

V. THEOREM. *The equation $Ax^2 + Hxy + By^2 + Gx + Fy + C = 0$ represents a central conic if $4AB - H^2 \neq 0$, and a non-central conic if $4AB - H^2 = 0$.*

Transform equation (2) to new axes OX', OY' parallel to the original axes, letting (x_0, y_0) represent the new origin referred to the original axes. The formulas of transformation are

$$x = x' + x_0, \qquad y = y' + y_0,$$

and substituting in (2) the result, after reduction, is

$$\left.\begin{aligned} &Ax'^2 + Hx'y' + By'^2 \\ &\quad + (2Ax_0 + Hy_0 + G)x' + (Hx_0 + 2By_0 + F)y' \\ &\quad + (Ax_0^2 + Hx_0y_0 + By_0^2 + Gx_0 + Fy_0 + C) = 0. \end{aligned}\right\} \tag{5}$$

This equation is to be reduced, if possible, to one without the terms in x' and y'. This requires that

$$2Ax_0 + Hy_0 + G = 0, \qquad Hx_0 + 2By_0 + F = 0. \tag{6}$$

Solving for x_0, y_0,

$$x_0 = \frac{HF - 2BG}{4AB - H^2}, \qquad y_0 = \frac{GH - 2AF}{4AB - H^2}. \tag{7}$$

These values of x_0 and y_0 are determinate if $4AB - H^2 \neq 0$, but if $4AB - H^2 = 0$ they are either indeterminate or infinite, according as the values of the numerators in (7) are zero or not. Hence if $4AB - H^2 \neq 0$ equations (7) determine a point, to which if the origin be transferred, equation (2) will reduce to a form having no terms of the first degree in x' and y'. It will therefore be the equation of a central conic with the new origin for center (see IV, above). But if $4AB - H^2 = 0$ no such determinate point exists, and hence the conic is non-central.

It should be carefully noted, as appears from (5), that when this transformation is made the coefficients of x'^2, $x'y'$, and y'^2 in the new equation are the same as those of the corresponding terms in the old equation, and that the new absolute term is obtained by substituting the coordinates of the new origin for x and y in the left-hand member of the original equation.

85. Transformed equation of a central conic.—By Art. 84, V, the equation of a central conic referred to coordinate axes passing through the center is in the form

$$Ax'^2 + Hx'y' + By'^2 + C' = 0, \tag{8}$$

where

$$C' = Ax_0^2 + Hx_0y_0 + By_0^2 + Gx_0 + Fy_0 + C, \qquad (9)$$

x_0 and y_0 having the values given by (7).

I. *To express the value of C' exclusively in terms of the coefficients A, H, B, G, F, C of equation* (2). From equation (9)

$$\begin{aligned} 2C' &= 2Ax_0^2 + 2Hx_0y_0 + 2By_0^2 + 2Gx_0 + 2Fy_0 + 2C, \\ &= x_0(2Ax_0 + Hy_0 + G) + y_0(Hx_0 + 2By_0 + F) \\ &\qquad + Gx_0 + Fy_0 + 2C. \end{aligned}$$

But by (6), the expressions in parentheses vanish,

$$\therefore \quad 2C' = Gx_0 + Fy_0 + 2C. \qquad (10)$$

Substituting in (10) the values of x_0, y_0 from (7) the result is

$$2C' = G\frac{HF - 2BG}{4AB - H^2} + F\frac{GH - 2AF}{4AB - H^2} + 2C,$$

or

$$C' = \frac{4ABC + FGH - AF^2 - BG^2 - CH^2}{4AB - H^2}. \qquad (11)$$

II. *The numerator of the fraction in* (11) *is called the discriminant of $Ax^2 + Hxy + By^2 + Gx + Fy + C$.*

Hence representing the discriminant by D, we have

$$D \equiv 4ABC + FGH - AF^2 - BG^2 - CH^2. \qquad (12)$$

EXERCISE. Show that of the six equations on p. 143, nos. 1, 2, 3, 5 represent central conics. Find the coordinates of the center of each of these four, and show that the equations reduce to the forms given below, when transformed to new axes through the center, parallel to the original axes.

1. Center at $(-2, -\frac{1}{2})$; $\quad 2x'^2 - 4x'y' + 5y'^2 - \frac{25}{4} = 0.$
2. " " $(-1, \frac{1}{2})$; $\quad 3x'^2 + 2x'y' - y'^2 + \frac{13}{4} = 0.$
3. " " $(-\frac{13}{5}, \frac{22}{5})$; $\quad x'^2 + 3x'y' + y'^2 + \frac{111}{5} = 0.$
5. " " $(\frac{6}{7}, \frac{20}{7})$; $\quad 6x'^2 - 5x'y' - y'^2 + 9 = 0.$

86. Equations of central conics. Final reduction.—When the general equation (2) represents a central conic it has been shown in Art. 84 how to reduce it to the form (8),

$$Ax'^2+Hx'y' + By'^2 + C' = 0 \qquad (i)$$

by transforming to new axes passing through the center.

We can next remove from (i) the term in xy by turning the axes through the angle determined by equation (4), p. 143, as shown in Art. 83. The equation is thus reduced to the form

$$A'x''^2 + B'y''^2 + C' = 0 \qquad (13)$$

where, as appears from (3), p. 143

$$\left.\begin{aligned} A' &= A\cos^2\theta + H\sin\theta\cos\theta + B\sin^2\theta, \\ B' &= A\sin^2\theta - H\sin\theta\cos\theta + B\cos^2\theta, \end{aligned}\right\} \qquad (14)$$

and C' remains unchanged.

I. *Selection of the value of* θ.—Since equation (4), p. 143, determines two values of 2θ less than 360°, which differ by 180°, there will always be one available value of 2θ between $-90°$ and $+90°$. It will be found convenient always to use this value of 2θ. Hence when $\tan 2\theta$ is positive, θ will be a positive angle between 0° and 45°; and when $\tan 2\theta$ is negative, θ will be a negative angle between 0° and 45°.

II. *Determination of the values of* A' *and* B'.—In (14) make the trigonometric substitutions

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta), \qquad \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta),$$

$$\sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta.$$

Rearranging the terms, the results are

$$\left.\begin{aligned} A' &= \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)\cos 2\theta + \tfrac{1}{2}H\sin 2\theta, \\ B' &= \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B)\cos 2\theta - \tfrac{1}{2}H\sin 2\theta. \end{aligned}\right\} \qquad (ii)$$

But $\tan 2\theta = H/(A - B)$, and hence from trigonometry

$$\left.\begin{aligned}\sin 2\theta &= \frac{H}{\pm\sqrt{H^2 + (A - B)^2}},\\ \cos 2\theta &= \frac{A - B}{\pm\sqrt{H^2 + (A - B)^2}}.\end{aligned}\right. \qquad (iii)$$

Substituting these values in (*ii*) the results, after reduction, are either

$$\left.\begin{aligned}A' &= \tfrac{1}{2}[A + B + \sqrt{H^2 + (A - B)^2},\\ B' &= \tfrac{1}{2}[A + B - \sqrt{H^2 + (A - B)^2},\end{aligned}\right\} \qquad (15a)$$

or

$$\left.\begin{aligned}A' &= \tfrac{1}{2}[A + B - \sqrt{H^2 + (A - B)^2},\\ B' &= \tfrac{1}{2}[A + B + \sqrt{H^2 + (A - B)^2},\end{aligned}\right\} \qquad (15b)$$

according as the sign of $\sqrt{H^2 + (A - B)^2}$ is taken as positive or negative.

To eliminate the ambiguity in the choice of values for A' and B', due to the double sign of $\sqrt{H^2 + (A - B)^2}$, it will be seen from I that, since 2θ is always to be taken in the first or fourth quadrants, $\cos 2\theta$ is always positive. Hence the sign of the radical is to be taken to agree with that of $(A - B)$. The following rule should therefore be used.

RULE. *If $A - B$ is positive use* (15*a*) *for determining A' and B', but if $A - B$ is negative use* (15*b*).

87. Discussion of equation (13), p. 147,

$$A'x''^2 + B'y''^2 + C' = 0.$$

It has been shown in Art. 86, II, how the values of A' and B' can be determined. Two other relations involving A' and B' are important The first is obtained by adding the two equations (14), which gives

$$A' + B' = A + B. \qquad (16)$$

Also, multiplying the equations (15*a*) or (15*b*) the result is

$$4A'B' = 4AB - H^2. \tag{17}$$

Since by hypothesis $4AB - H^2 \neq 0$, equation (17) shows that neither A' nor B' can be zero. Hence in discussing equation (13) there are only two cases to consider, according as C' is zero or not. The latter will be taken first.

I. $C' \neq 0$. Equation (13) can then be put in the form

$$A'x''^2 + B'y''^2 = -C',$$

or

$$\frac{x''^2}{-\frac{C'}{A'}} + \frac{y''^2}{-\frac{C'}{B'}} = 1. \tag{18}$$

If $-(C'/A')$ and $-(C'/B')$ are both positive, (18) represents an ellipse. This will be the case when A' and B' have like signs, and C' the opposite sign.

If $-(C'/A')$, and $-(C'/B')$ have opposite signs, (18) represents a hyperbola. This will be the case when A' and B' have unlike signs.

If A', B', C' all have the same sign, equation (18) represents an imaginary locus, called an imaginary ellipse, since the equation is in the ellipse form.

From equation (11), p. 146, $C' = D/(4AB - H^2)$, where D is the discriminant (12). Hence when $C' \neq 0$, it follows that $D \neq 0$.

This discussion shows that equation (13), when $C' \neq 0$, represents an ellipse, real or imaginary, or a hyperbola, according as the product $A'B'$ is positive or negative. Hence from (17), it follows that the general equation (2) represents

$$\left.\begin{array}{l}\textbf{an ellipse,} \text{ when } 4AB - H^2 > 0, \text{ and } D \neq 0,\\ \textbf{a hyperbola,} \text{ when } 4AB - H^2 < 0, \text{ and } D \neq 0.\end{array}\right\} \tag{19}$$

II. $C' = 0$. When $C' = 0$ equation (13) takes the form

$$A'x''^2 + B'y''^2 = 0. \tag{20}$$

If A' and B' have unlike signs the left-hand member of this equation can be factored in real factors of the first degree in x'' and y''. Hence it represents a pair of real intersecting lines meeting at the origin. Also from (17) $4AB - H^2 < 0$.

If A' and B' have like signs the only real values of x'' and y'' which will satisfy the equation are $x'' = 0$, $y'' = 0$. Hence the equation may be regarded as that of a *point ellipse*, the limit to which the locus of equation (13) approaches when A' and B' have like signs and C' approaches the limit zero. Or, since in this case the left-hand member of equation (20) can be factored in imaginary factors of the first degree, the equation may be said to represent a pair of imaginary lines intersecting at a real point, the origin. In this case from (17) $4AB - H^2 > 0$.

When $C' = 0$, $D = 0$ also, as appears from (11). Hence equation (2) represents

$$\left.\begin{array}{l} \textbf{real intersecting lines, } \text{when} \\ \qquad\qquad 4AB - H^2 < 0, \text{ and } D = 0, \\ \textbf{imaginary intersecting lines, } \text{or a } \textbf{point ellipse, } \text{when} \\ \qquad\qquad 4AB - H^2 > 0, \text{ and } D = 0. \end{array}\right\} \tag{21}$$

EXAMPLES. 1. Discuss fully the equation

$$3x^2 + 2xy - y^2 + 5x + 3y + 5 = 0.$$

The results already found for this equation in Ex. 2, p. 143 and p. 146, will be used in this discussion.

First step. $4AB - H^2 = -16$, and by (12) p. 146, $D = -52$. Hence by (19) the locus is a hyperbola.

Second step. Determine the center by (7), p. 145, and transform the equation to new axes through this point, parallel to the axes XOY. This is done by using (8) and (10), pp. 145, 146. On p. 146 the center

has been found to be $(-1, \frac{1}{2})$, and the transformed equation is

$$3x'^2 + 2x'y' - y'^2 + \tfrac{13}{4} = 0.$$

The coordinate axes are now $X'O'Y'$.

Third step. By (4) $\theta = \frac{1}{2} \tan^{-1} \frac{1}{2}$. Turning the axes through this angle they take the position $X''O'Y''$. The equation now takes the form (13) where, since $A - B$ is positive, the values of A' and B' are determined from (15*a*). Thus

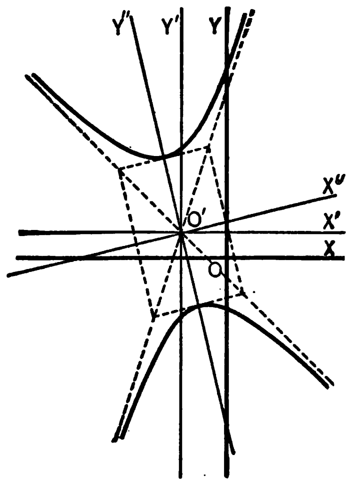


FIG. 77

$$A' = \tfrac{1}{2}[2 + \sqrt{4+16}] = 1 + \sqrt{5},$$

$$B' = \tfrac{1}{2}[2 - \sqrt{4+16}] = 1 - \sqrt{5}.$$

Hence the transformed equation is

$$(1 + \sqrt{5})x''^2 + (1 - \sqrt{5})y''^2 + \tfrac{13}{4} = 0,$$

or
$$(\sqrt{5} + 1)x''^2 - (\sqrt{5} - 1)y''^2 = -\tfrac{13}{4}.$$

This reduces to

$$\frac{x''^2}{\dfrac{13}{4(1+\sqrt{5})}} - \frac{y''^2}{\dfrac{13}{4(\sqrt{5}-1)}} = -1.$$

It is now in the form corresponding to (21), p. 92, and therefore represents a hyperbola with transverse axis in the Y-axis.

The semi-transverse axis $= \sqrt{\dfrac{13}{4(\sqrt{5}-1)}} = \sqrt{\dfrac{13}{16}(\sqrt{5}+1)} = 1.62,$

and the semi-conjugate axis $= \sqrt{\dfrac{13}{4(\sqrt{5}+1)}} = \sqrt{\dfrac{13}{16}(\sqrt{5}-1)} = 1.00.$

The curve can now be drawn as shown in the figure.

2. Discuss the equation $3x^2 - 8xy + 4y^2 - 2x - 4y - 8 = 0$.

First step. $4AB - H^2 = -16$, $D = 0$. Hence by (21) the locus is a pair of intersecting real lines.

Second step. The coordinates of the center (the intersection of the lines) are found by (7) to be $x_0 = -3$, $y_0 = -\frac{5}{2}$; and hence the equation, referred to the new axes $X'O'Y'$, with the point $(-3, -\frac{5}{2})$ as the new origin, is by (8) and (11)

$$3x'^2 - 8x'y' + 4y'^2 = 0.$$

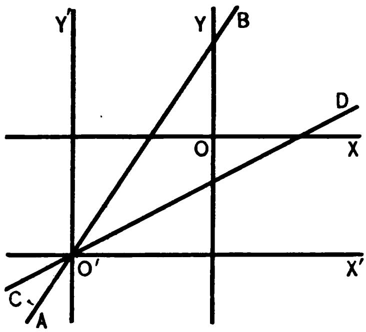


FIG. 78

Third step. The left-hand side of this equation is a homogeneous expression of the second degree in x' and y'. Such an expression can always be separated into linear factors. See III, p. viii. The factors in this case are $3x' - 2y'$, and $x' - 2y'$. Hence the two real lines which the given equation represents are

$$3x' - 2y' = 0,$$

and

$$x' - 2y' = 0,$$

the lines AB, CD, respectively in the figure.

NOTE. Since every equation in the form (8), p. 145, in which $C' = 0$, can be treated as in the "*Third step*" above, it is unnecessary with such an equation to revolve the axes to remove the term in xy.

3. Discuss fully the equation of exercise 1, p. 143, making use of the results given on pp. 143 and 146, and draw the curve.

4. In the same manner discuss the equation of exercise 3, p. 143.

5. Discuss also the equation of exercise 5, p. 143.

88. Equations of non-central conics.—The general equation

$$Ax^2 + Hxy + By^2 + Gx + Fy + C = 0 \qquad (i)$$

represents a non-central conic when $4AB - H^2 = 0$, as has been shown Art. 84, V. If $H = 0$ then either A or B must also equal zero, and no further investigation is necessary, as the equation can be treated as in Art. 64, Ex. 2. In the general case, when H is not zero, A and B must have the same sign, and hence may

always be taken as positive. The condition $4AB - H^2 = 0$ also shows that the first three terms in (i) form a perfect square.

To simplify the subsequent discussion equation (i) is multiplied by $4B$, and $4AB$ the coefficient of x^2 is replaced by its equal H^2. The equation may then be written

$$(Hx + 2By)^2 + 4B(Gx + Fy) + 4BC = 0. \qquad (ii)$$

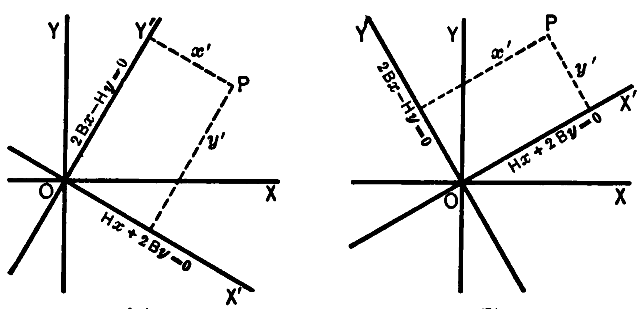
Equation (ii) is now transformed to new axes, taking

$$Hx + 2By = 0, \quad \text{and} \quad 2Bx - Hy = 0, \qquad (iii)$$

for the new X- and Y-axes respectively. These are seen to be perpendicular to each other by (17), p. 52. The slope of the first is $-(H/2B)$, hence this change of axes is equivalent to turning the original axes through the angle

$$\theta = \tan^{-1}\left(-\frac{H}{2B}\right).$$

Since B is positive; θ will be a negative angle, between 0° and − 90° in value, if H is positive, and a positive angle, between 0° and 90° in value, if H is negative.



(a) H positive. (b) H negative.

FIG. 79

The new abscissa x' of any point P is the perpendicular distance from OY' to P, and the new ordinate y' is the perpendicular distance from OX' to P. Therefore by (3), p. 56

$$x' = \frac{2Bx - Hy}{\sqrt{4B^2 + H^2}}, \qquad y' = \frac{Hx + 2By}{\sqrt{4B^2 + H^2}}, \tag{iv}$$

where the sign of the radical is to be taken positive.

Note. For every point on OX', Fig. 79 (a), the original abscissa x is positive, and the original ordinate y is negative. Therefore since both B and H are positive the new abscissa of every point on OX' is positive, as appears from the value of x' given in (iv). Hence OX' is the positive portion of the new X-axis. Similarly it can be shown that OY' is the positive portion of the new Y-axis in the same figure, and that in Fig. 79 (b), where H is negative, points in the quadrant $X'OY'$ have both coordinates positive.

Solving equations (iv) for x and y, we have

$$x = \frac{2Bx' + Hy'}{\sqrt{4B^2 + H^2}}, \qquad y = \frac{-Hx' + 2By'}{\sqrt{4B^2 + H^2}}. \tag{v}$$

Substituting from (v) in (ii) and reducing

$$(4B^2 + H^2)y'^2 + \frac{4B}{\sqrt{4B^2 + H^2}}[(2BG - HF)x' + (GH + 2BF)y'] + 4BC = 0. \tag{vi}$$

Restoring to H^2 its value $4AB$, and dividing the equation by $4B$, the result is

$$(A + B)y'^2 + \frac{(2BG - HF)}{2\sqrt{B^2 + AB}}x' + \frac{(2BF + GH)}{2\sqrt{B^2 + AB}}y' + C = 0. \tag{vii}$$

This equation is now in the form

$$B'y'^2 + G'x' + F'y' + C = 0, \tag{22}$$

where

$$B' = A + B,$$

$$G' = \frac{2BG - HF}{2\sqrt{B^2 + AB}}, \qquad F' = \frac{2BF + GH}{2\sqrt{B^2 + AB}}. \tag{23}$$

By the process used in Art. 64, Ex. 2, equation (22) is easily reduced to the form

$$\left(y' + \frac{F'}{2B'}\right)^2 = -\frac{G'}{B'}\left(x - \frac{F'^2 - 4B'C}{4B'G'}\right), \tag{24}$$

provided G' is not zero. This equation represents a parabola, with principal axis parallel to the X-axis, and vertex at the point $((F'^2-4B'C)/4B'G', -(F'/2B'))$. See equation (29), p. 99. The length of the latus rectum is $2p. = |G'/B'|$.

89. Degenerate non-central conics.

I. If $G' = 0$, equation (22) takes the form

$$B'y'^2 + F'y' + C = 0. \tag{25}$$

Multiplying by $4B'$, and adding and subtracting F'^2 equation, (25) takes the form

$$(2B'y' + F')^2 - (F'^2 - 4B'C) = 0,$$

or factoring,

$$(2B'y' + F' + \sqrt{F'^2 - 4B'C})(2B'y' + F' - \sqrt{F'^2 - 4B'C}) = 0.$$

Equation (25) therefore represents the two straight lines,

$$\left.\begin{array}{l} 2B'y' + F' + \sqrt{F'^2 - 4B'C} = 0 \\ 2B'y' + F' - \sqrt{F'^2 - 4B'C} = 0 \end{array}\right\} \tag{26}$$

parallel to the X-axis. These lines will be real and distinct, coincident, or imaginary according as $F'^2 - 4B'C >$, $=$, or < 0.

II. *When* $4AB - H^2 = 0$, *the condition* $G' = 0$ *in* (22) *is equivalent, in the general equation* (2), *to* $D = 0$. See (12), p. 146.

In the last term of the expression for D, substitute for H^2 its

value $4AB$, then

$$D = FGH - AF^2 - BG^2. \qquad (i)$$

Also, from (23), if $G' = 0$,

$$2BG - HF = 0, \qquad \text{or} \qquad F = \frac{2BG}{H}. \qquad (ii)$$

Substituting this value of F in (i) the result is

$$D = 2BG^2 - 4\frac{AB^2G^2}{H^2} - BG^2,$$

or, since by hypothesis $H^2 = 4AB$,

$$D = 2BG^2 - BG^2 - BG^2 = 0. \qquad (iii)$$

III. *Conclusion.* The general equation (2) represents

$$\left.\begin{array}{l} \textbf{a parabola}, \text{ when } 4AB - H^2 = 0 \text{ and } D \neq 0, \\ \textbf{parallel} \text{ or } \textbf{coincident lines}, \text{ when } 4AB - H^2 = 0 \\ \qquad\qquad\qquad\qquad \text{and } \qquad\qquad D = 0. \end{array}\right\} \qquad (27)$$

In the latter case the lines are real and distinct, coincident, or imaginary according as

$$F'^2 - 4B'C >, \ =, \ < 0. \qquad (28)$$

EXAMPLES. 1. Determine the nature and position of the locus of the equation $4x^2 - 4xy + y^2 + 2x - 6y - 10 = 0$.

First step. $4AB - H^2 = 0$, $D = -100$. Hence by (27) the locus is a parabola.

Second step. By (iii), p. 153, the lines

$$-4x + 2y = 0, \qquad 2x + 4y = 0$$

are taken as new axes of coordinates, and hence by (v), p. 154, the formulas of transformation are

$$x = \frac{x' - 2y'}{\sqrt{5}}, \qquad y = \frac{2x' + y'}{\sqrt{5}}. \qquad (i)$$

Substituting in the given equation and reducing, the result is

$$y'^2 - \frac{2}{\sqrt{5}}(x' + y') - 2 = 0,$$

which is the equation of the parabola referred to the axes $X'OY'$.

Third step. The last equation is easily transformed as in Art. 64, Ex. 2 into

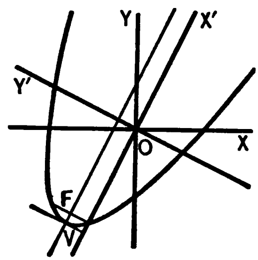


FIG. 80

$$\left(y' - \frac{1}{\sqrt{5}}\right)^2 = \frac{2}{\sqrt{5}}\left(x' + \frac{11}{2\sqrt{5}}\right),$$

showing that the vertex is at the point

$$\left(-\frac{11}{2\sqrt{5}},\ \frac{1}{\sqrt{5}}\right) = (-2.46,\ 0.45),\ (ii)$$

The focus is at a distance

$$\frac{1}{4}\left(\frac{2}{\sqrt{5}}\right) = \frac{1}{2\sqrt{5}}$$

on the positive side of the vertex, that is at the point

$$\left(-\frac{11}{2\sqrt{5}} + \frac{1}{2\sqrt{5}},\ \frac{1}{\sqrt{5}}\right) = (-2.24,\ 0.45). \qquad (iii)$$

To determine the position of vertex and focus referred to the original axes substitute from (*ii*) and (*iii*) in (*i*). Thus it is found that the vertex is at $(-\frac{3}{2}, -2)$ and the focus at $(-\frac{7}{5}, -\frac{9}{5})$. The slope of the axis is 2.

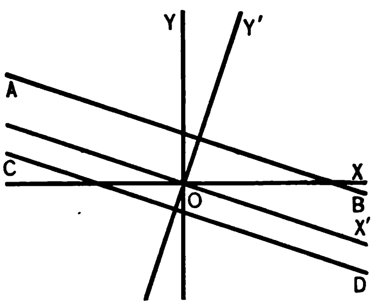


FIG. 81

2. Investigate the equation $x^2 + 6xy + 9y^2 - 2x - 6y - 10 = 0$.

First step. $4AB - H^2 = 0$, and $D = 0$. Hence by (27) the locus is a pair of parallel or coincident lines.

By (23) $F' = -(20/\sqrt{10}) = -2\sqrt{10}$, $B' = 10$, and $F'^2 - 4B'C = 440$. By (28) the lines are therefore real and distinct.

Second step. By (*iii*), p. 153, the new X- and Y-axes are respectively $x + 3y = 0$, and $3x - y = 0$. The transformed equation can be written at once from (25), p. 155.

$$10y'^2 - 2\sqrt{10}y' - 10 = 0.$$

Solving this equation for y',

$$y' = \frac{1 \pm \sqrt{11}}{\sqrt{10}}, \text{ or } y' = 1.36 \text{ and } -0.73.$$

The two lines are shown in Fig. 81 in the positions AB and CD.

90. Summary.—The results obtained in this Chapter are summarized in the following table:

	$D \neq 0$	$D = 0$
$4AB - H^2 > 0$	**Real ellipse** if C' differs in sign from A' and B'. Arts. 85, 86, 87, I. **Imaginary ellipse** if A', B', C' have the same sign. Arts. 85, 86, and 87, I.	**A pair of imaginary lines** meeting at the real point given by (7), p. 145. See Art. 87, II.
$4AB - H^2 < 0$	**Hyperbola.** Arts. 85, 86, and 87, I.	**A pair of real lines** meeting at the point given by (7), p. 145. See Art. 87, II.
$4AB - H^2 = 0$	**Parabola.** Art. 88.	**Real parallel lines,** when $F'^2 - 4B'C > 0$. **Coincident lines,** when $F'^2 - 4B'C = 0$. **Imaginary lines,** when $F'^2 - 4B'C < 0$. See (23), p. 155, and Art. 89.

The procedure in treating an individual equation has been illustrated in the exercises worked. The steps may be summarized as follows:

A. Compute $4AB - H^2$, and D (12), p. 146. The results will determine the species of the conic, and whether it is proper or degenerate. See (19), p. 149, (21), p. 150, and (27), p. 156.

B. (i) If $4AB - H^2 \neq 0$, find the coordinates of the center by (7), p. 145, and transform the equation to new axes through this point parallel to the original axes. See Art. 85.

(ii) If $D \neq 0$, turn the axes through the angle

$$\theta = \tfrac{1}{2} \tan^{-1} \frac{H}{A - B},$$

and the equation will then be in the form (13), p. 147. See Art. 86.

(iii) If $D = 0$ the first transformation will reduce the equation to a factorable form, and further transformation is unnecessary.

C. (i) If $4AB - H^2 = 0$, choose new axes as stated in Art. 88, and transform the equation to these axes.

(ii) If $D \neq 0$ the new equation will be in the form (22), p. 154, and the parabola can be constructed from this equation; or by a second transformation the origin can be moved to the vertex, the axes remaining parallel to the second set.

(iii) If $D = 0$, the equation after the first transformation will contain only one variable. It can therefore be readily factored, so that further transformation is unnecessary.

EXERCISES ON CHAPTER VIII

Normal Exercises

In exercises 1–12 determine the species of the conic whose equation is given, and its location with reference to the axes. In the case of each central conic determine the position of the center, the slope of the principal axis, and, if the conic is real and not degenerate, the lengths of the semi-axes. If the conic is non-central and not degenerate determine the position of the vertex, the slope of the axis, and the value of p. Draw each locus.

1. $5x^2 - 5xy + 3y^2 - 5x + 6y - 3 = 0.$

   Ans. A real ellipse, center at $(0, -1)$, slope of major axis 1.48, semi-axes 2.14, 0.95.

2. $5x^2 + 2xy - y^2 + 6x - 4y - 12 = 0.$

   Ans. A hyperbola, center at $(-\frac{1}{6}, -\frac{13}{6})$, slope of transverse axis 0.16, semi-axes 1.26, 2.65.

3. $9x^2 + 12xy + 4y^2 + 10x - 54y - 68 = 0.$

Ans. A parabola, vertex $(\frac{20}{13}, -\frac{21}{26})$, $p = \frac{7}{13}\sqrt{13}$.

4. $x^2 + 6xy + 9y^2 + 2x + 6y - 25 = 0.$

Ans. Parallel lines with slope $-\frac{1}{3}$.

5. $2x^2 + 6xy + 5y^2 - 10x - 6y + 53 = 0.$

Ans. Imaginary lines, meeting at $(16, -9)$.

6. $9x^2 - 6xy + y^2 + 12x - 4y + 4 = 0.$

7. $3x^2 - 6xy + 5y^2 + 5x - 10y + 12 = 0.$

8. $x^2 - 4xy + 3y^2 + 3x - 7y + 2 = 0.$

9. $3x^2 + 6xy + 4y^2 + 4x - 8y - 12 = 0.$

10. $3x^2 - 2xy - 5y^2 + 3x + 7y - 8 = 0.$

11. $4x^2 - 4xy + y^2 - 4x + 2y + 4 = 0.$

12. $3x^2 - 12xy + 2x - 3y - 20 = 0.$

General Exercises

13. Prove that $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is a parabola which touches the coordinate axes at the extremities of the latus rectum.

14. Prove that $\sqrt{bx} + \sqrt{ay} = \sqrt{ab}$ is a parabola which touches the X-axis at $x = a$, and the Y-axis at $y = b$.

15. For what value of c will $x^2 + 2xy - y^2 + 6x - 2y + c = 0$ represent a pair of real lines? Ans. $c = 1$.

16. For what value of c will $2xy - 4x + 7y + c = 0$ represent a pair of real lines? Ans. $c = -14$.

17. For what value of c will $x^2 + 4xy + 4y^2 + 4x + 8y + c = 0$ represent a pair of coincident lines? Ans. $c = 4$.

18. For what value of c will $a^2x^2 + 2abxy + b^2y^2 + c = 0$ represent a pair of coincident lines?

19. Prove that $hxy + gx + fy + c = 0$ is in general an equilateral hyperbola with $hx + f = 0$, and $hy + g = 0$ for asymptotes, but that if $fg = ch$ it represents a pair of real lines.

20. If in the general equation (2), p. 142, $4AB - H^2 = 0$, and if the term in xy be removed by turning the axes through the angle θ [Eq. (4), p. 143], show that the resulting equation will always be one in which the coefficient of either x^2 or y^2 is zero.

CHAPTER IX

HIGHER PLANE CURVES, PARAMETRIC EQUATIONS

91. Introductory remarks.

DEFINITION. *The locus corresponding to any equation not in the form* $Ax + By + C = 0$, *or* $Ax^2 + Hxy + By^2 + Gx + Fy + C = 0$, *and not reducible to one of these two forms, is called a* ***higher plane curve.***

For example, $y^3 = cx^2$, $x^3 + y^3 + 3axy = 0$, $y = a \sin bx$, $y = \log x$, $y = \tan^{-1} x$, $y = e^{ax}$ are all equations of higher plane curves. To determine whether an equation containing radicals or fractional exponents, is the equation of a higher plane curve, it must first be reduced to a form without radicals or fractional exponents. A few curves of this kind will be discussed.

92. Loci of equations of the form $x^n y = c$.—The curves corresponding to equations of the form

$$x^n y = c, \tag{1}$$

where c is a constant, and n has any numerical value, constitute an important family. Equations of this form arise in the study of the expansion of gases and vapors, and the corresponding curves are used to represent graphically the changes of pressure and volume which take place under different conditions, as for example, in the cylinder of a gas or steam engine.

I. For certain values of n equation (1) represents a straight line or a conic. Thus, for

$n = 0$ it is the equation of the straight line $y = c$,
$n = 1$ it is the equation of the hyperbola $xy = c$,
$n = -\frac{1}{2}$ it is the equation of the parabola $y^2 = c^2x$,
$n = -1$ it is the equation of the straight line $y = cx$.
$n = -2$ it is the equation of the parabola $y = cx^2$.

For other values of n the locus is a higher plane curve.

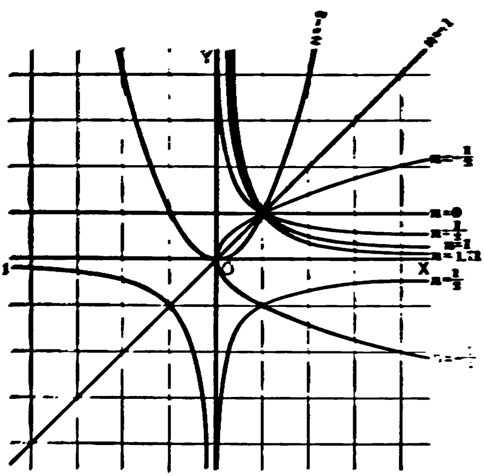
Two important general conclusions with reference to equation (1), may be noted.

II. *For any given value of n, c may be determined so that the locus shall pass through any one given point in the plane.*

For example, let $n = \frac{3}{2}$, then the locus of $x^{\frac{3}{2}}y = c$ will pass through the point (4, 3) if $4^{\frac{3}{2}} \cdot 3 = c$, that is if $c = 24$.

III. *For all values of n the locus will pass through the point (1, c).*

For equation (1) is always satisfied if $x = 1$, $y = c$.



$x^n y = 1$

Fig. 82

In Fig. 82 are drawn several curves of this family exhibiting the shape and position of the curves for different values of n. The value $c = 1$ is used in all the curves. Hence they all pass through the point (1, 1).

IV. In connection with the curves discussed in this article a device frequently employed is to use what may be called logarithmic plotting. If the logarithms of both sides of equation

(1) be taken the result is

$$n \log x + \log y = \log c. \tag{i}$$

This equation is of the first degree in $\log x$ and $\log y$. Hence if $\log x$ and $\log y$ be plotted as coordinates instead of x and y the *resulting figure is a straight line*, with slope $-n$ and y-intercept $\log c$. Therefore to each curve of the family $x^n y = c$ there corresponds a straight line of the family (i) with the same value of n.

In applying equation (1) to the study of problems involving pressures and volumes of gases and vapors this method is conveniently employed. All of the properties derivable from the curves themselves can be obtained equally well from the corresponding straight lines in the logarithmic diagram.

Paper ruled logarithmically is made for this purpose. The lines on the paper are spaced proportionately to the logarithms of the natural numbers instead of the numbers themselves. By its use the transformation from the actual values of x and y to the logarithmic values is made mechanically without any reference to tables of logarithms.

It should be noted that this transformation of these curves into straight lines applies only to those portions which lie in the first quadrant, where x and y are both positive.

NOTE. The equations $x^n y = c$, and $y = cx^n$ are essentially the same; as by changing the sign of the exponent the one form passes into the other. Irrational values of the exponent n are sometimes met with. In such cases an approximate rational value of the exponent is used. For example if $n = \sqrt{2}$, the approximate value $n = 1.41$ may be used.

93. The sine and cosine curves.

I. *The sine curve or sinusoid.* The equation of this curve is

$$y = \sin x. \tag{2}$$

In plotting this curve the values of x are expressed in radians,

so that graphically x will be the length of the arc, radius $= 1$, which subtends the angle whose sine is y. In this way both x and y are expressed in terms of the same unit of length, as shown in trigonometry. The same remark applies to all equations in which either of the variables stands for an angle. The following table gives a few pairs of simultaneous values of x and y.

$x = 0$	$\frac{1}{6}\pi = 0.52$	$\frac{1}{4}\pi = 0.79$	$\frac{1}{2}\pi = 1.57$	$\frac{3}{2}\pi = 4.71$	etc.
$y = 0$	$\frac{1}{2} = 0.50$	$\frac{1}{2}\sqrt{2} = 0.71$	1	-1	etc.

For an accurate construction of the curve a table of natural sines should be used. In the figure the solid curve is the sine

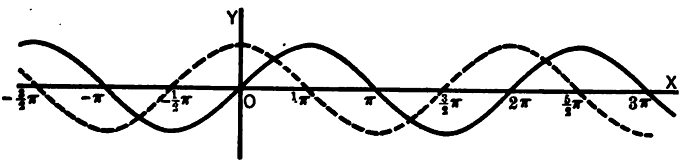


Fig. 83

curve. It extends indefinitely to right and left along the X-axis, and cuts this axis at an infinite number of points, whose abscissas are $x = n\pi$, where n is any whole number, positive or negative. The value of y is never greater than $+ 1$, nor less than $- 1$.

II. The Cosine Curve. The equation of this curve is

$$y = \cos x. \tag{3}$$

This curve is constructed in the same manner as the sine curve. It is the same curve shifted to the right or left along the X-axis a distance $\frac{1}{2}\pi$. It is drawn dotted in Fig. 83.

94. Modified forms of the sine curve.—The curve whose equation is (2) is the fundamental form of the sine curve. Some modifications of it are important from their application in the theory of wave motion.

$$y = a \sin x. \tag{4}$$

The curves which correspond to equations of this form for different numerical values of the constant a, differ from the locus of equation (2) only in the fact that the highest and lowest points on the curve are at distances $+a$ and $-a$ respectively from the X-axis, and the ordinate of any other point on any of the curves is to the corresponding ordinate on the curve of (2) as $a : 1$. They all cut the X-axis where $\sin x = 0$, that is where $x = n\pi$, n being any whole number, positive or negative.

II. $$y = a \sin bx. \qquad (5)$$

In this equation a and b have any constant numerical value. The effect of the constant a has been pointed out in the discussion of equation (4) in I. The value of the constant b affects the location of the points where the curve cuts the X-axis, for these points occur where $\sin bx = 0$, which gives $bx = n\pi$, or $x = (n/b)\pi$, where n is any whole number, positive or negative. Hence for $b > 1$ these points are nearer each other than in the curve of (2) or (4), and for $b < 1$, they are farther apart.

Taking $a = \frac{3}{2}$, $b = 2$, the equation is $y = \frac{3}{2} \sin 2x$, and the corresponding curve is Fig. 84a. If $a = \frac{3}{2}$, $b = \frac{1}{3}$, the equation is $y = \frac{3}{2} \sin \frac{1}{3}x$, and the corresponding curve is Fig. 84b.

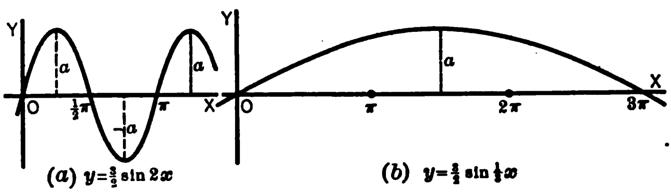


FIG. 84

III. $$y = a \sin (bx + c). \qquad (6)$$

The curve corresponding to this equation, in which c, as well as a and b, has any numerical value, differs from that of equation (5) only in the location of the points in which it cuts

the X-axis. Let $a = \frac{3}{2}$, $b = 2$, $c = 0.52$, then the equation is

$$y = \tfrac{3}{2} \sin (2x + 0.52). \qquad (i)$$

To determine where the curve cuts the X-axis let $y = 0$, then

$$2x + 0.52 = n\pi, \qquad \text{or} \qquad x = \tfrac{1}{2}(n\pi - 0.52). \qquad (ii)$$

Hence every integral value of n in (ii) determines the abscissa of a point where the curve cuts the X-axis. Thus:

$$\begin{aligned} n &= -1 \text{ gives } x = -\tfrac{1}{2}\pi - 0.26 = -1.83, \\ n &= 0 \text{ gives } x = -0.26, \\ n &= 1 \text{ gives } x = \tfrac{1}{2}\pi - 0.26 = 1.31, \\ n &= 2 \text{ gives } x = \pi - 0.26 = 2.88, \\ &\text{etc.} \qquad \text{etc.} \qquad \text{etc.} \end{aligned}$$

The curve is therefore the same as that in Fig. 84a moved a distance 0.26 to the left, as shown in Fig. 85, where

$$OA = -0.26.$$

In order to determine the value of y corresponding to any given value of x, proceed as follows. Let $x = 0.40$, then $2x + 0.52 = 1.32$, then

$$\begin{aligned} y &= \tfrac{3}{2} \sin (1.32 \textit{ radians}) \\ &= \tfrac{3}{2} \sin 75^\circ 38' = \tfrac{3}{2}(0.969) = 1.45. \end{aligned}$$

Hence (0.40, 1.45) is a point on the curve.

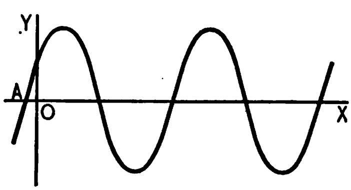


FIG. 85

From this it may be concluded in general that the curve of $y = a \sin (bx + c)$ cuts the X-axis at points determined by $bx + c = n\pi$, or $x = (n\pi - c)/b$, where n is any integer, and hence that the curve differs from that of $y = a \sin bx$ only in being shifted a distance $-(c/b)$ along the X-axis.

95. Logarithmic curves.

I. The curve

$$y = \log_a x. \qquad (7)$$

The general shape of this curve is the same for all values of the base a, provided a is positive and greater than unity, the only cases which are usually met with. For all such values of a we have $\log a = 1$, $\log 1 = 0$, and $\log 0 = -\infty$. Hence the corresponding curves all cut the X-axis at $x = 1$, they all approach the negative end of the Y-axis without cutting it, and as x increases from $x = 1$, y also increases, but more slowly

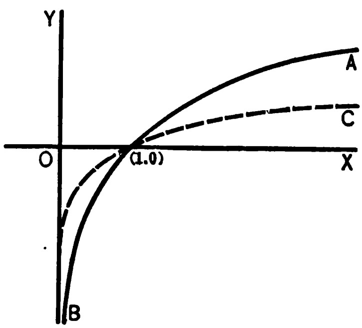


FIG. 86

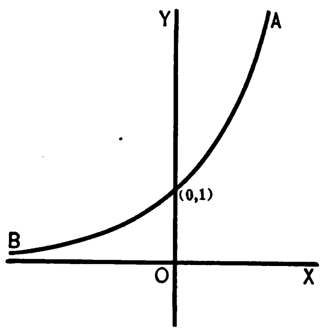


FIG. 87

than x. In Fig. 86 the curve drawn full is the locus of $y = \log_e x$ and the dotted curve is the locus of $y = \log_{10} x$.

II. The curve

$$y = e^x. \qquad (8)$$

This curve, Fig. 87, is the logarithmic curve in a different position, as $y = e^x$, and $x = \log_e y$ are the same equation in different forms. When $x = 0$, $y = 1$. As x increases, y increases also, and as x approaches $-\infty$, y approaches zero.

96. $$y = e^{ax} \sin bx. \qquad (9)$$

This equation and the corresponding curve are practically

important on account of their connection with the theory of alternating currents. The letters a and b stand for numerical constants, and e is the Napierian base, $2.718\cdots$. (V (c), p. ix.)

For all finite values of a and b, $y = 0$ when $x = 0$, hence all curves of this family pass through the origin.

Since e^{ax} cannot vanish for any finite value of x, the points where the curve cuts the X-axis are determined by making $\sin bx = 0$, from which, as in Art. 94, $bx = n\pi$, or $x = (n/b)\pi$, where n has in turn all integral values. These points are therefore the same as for the curves of equation (5). Also, since $\sin bx$ cannot be numerically greater than unity, the curve lies between the two curves $y = \pm e^{ax}$, touching $y = e^{ax}$ when $\sin bx = 1$, and $y = -e^{ax}$ when $\sin bx = -1$.

Let $a = \frac{3}{2}$, $b = 6$, then the equation is

$$y = e^{\frac{3}{2}x} \sin 6x. \qquad (i)$$

This curve will cut the X-axis at the points determined by $x = \frac{1}{6}n\pi$, where n is any integer, that is at $x = 0, \frac{1}{6}\pi, \frac{2}{6}\pi, \frac{3}{6}\pi$, etc.

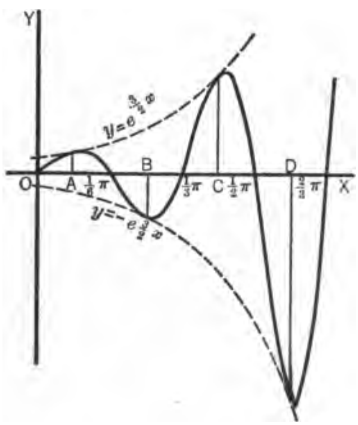


FIG. 88

Let $x = 0.2$ in (i), then

$$y = e^{0.3} \sin (1.2 \text{ radians})$$
$$= e^{0.3} \sin 68° \, 45'.$$

Using logarithms the result is $\log y = 0.09971$, or $y = 1.26$. The ordinates of other points are similarly computed.

A portion of the curve on the positive side of the origin is drawn in Fig. 88. The horizontal unit used in the drawing is ten times the vertical unit.

The points A, B, C, D, etc., midway between the points where the curve cuts OX, are the feet of the ordinates to the points where the curve touches the two curves $y = \pm e^{\frac{1}{2}x}$.

EXERCISES. 1. Construct on a diagram similar to Fig. 82 the loci of the family $x^n y = c$ for which $c = 1$, and $n = \frac{1}{3}, 3, -\frac{1}{3}, -3$ respectively.

2. Construct on one diagram the curves of the family $x^n y = c$, for which $c = 2$, and $n = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$ respectively. Point out the general differences between these curves and those of Fig. 82.

3. Name some characteristics common to all curves of the family $x^n y = 1$, (*i*) in which the value of n is positive, (*ii*) in which the value of n is between 0 and -1, (*iii*) in which the value of n is less than -1.

4. Construct on one diagram the loci whose equations are $y = \tan x$, $y = 4 \tan x$, $y = \tan 4x$, $y = \tan (x + 4)$.

5. State in what definite way each of the loci $y = a \tan x$, $y = \tan bx$, $y = \tan (x + c)$, $y = a \tan (bx + c)$ differs from the curve $y = \tan x$.

6. What is the effect on the graph of $y = a \tan (bx + c)$ of a change in a? in b? in c?

7. Construct the following loci

(*a*) $y = \sin 2x$,	(*b*) $y = \sin \pi x$,	(*c*) $y = \sin^2 x$,
(*d*) $y = x + \sin x$,	(*e*) $y = 1 - \cos x$,	(*f*) $y = \tan^2 x$,
(*g*) $y = \sin^{-1} x$,	(*h*) $y = \tan^{-1} x$,	(*i*) $y = 4 \sin^{-1} (x^2)$,
(*j*) $y = x \sin^{-1} x$,	(*k*) $y = x + \sin^{-1} x$,	(*l*) $y = e^x \sin 2x$,
(*m*) $y = 4e^x$,	(*n*) $y = e^{x+4}$,	(*o*) $y = 4 \log x$,
(*p*) $y = \log (x + 4)$,	(*q*) $y = e^x \log x$,	(*r*) $y = x \log x$,

97. Parametric equations.—When the coordinates of points on a curve are expressed separately in terms of a third variable, or parameter, the two equations are called the **parametric equations** of the curve.

This was done in the case of the ellipse, Art. 78, where the coordinates of any point on the ellipse are expressed in equations (5), p. 133, in terms of the eccentric angle ϕ,

$$x = a \cos \phi, \qquad y = b \sin \phi. \tag{i}$$

When this can be done it is merely an alternative to that of expressing the relation between x and y by means of a single equation. The latter can be derived from the parametric equations by combining them so as to eliminate the parameter, as shown in Art. 81.

Thus eliminating ϕ between the two equations (i) by writing them in the form

$$\frac{x}{a} = \cos\phi, \qquad \frac{y}{b} = \sin\phi,$$

and then squaring, and adding, the result is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the ordinary equation of the ellipse.

EXAMPLES. 1. Show that $x = a \sec t$, $y = b \tan t$ are parametric equations of the hyperbola.

The two given equations can be expressed in the form

$$\frac{x}{a} = \sec t, \qquad \frac{y}{b} = \tan t.$$

Hence, squaring and subtracting the second from the first

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is the equation of a hyperbola.

2. Construct the locus whose parametric equations are $x = \frac{1}{4}t^2 - 1$, $y = t + 2$.

To determine the coordinates of points on the locus assign values to the parameter t, and compute the corresponding pairs of values of x and y. Thus

$t =$	-4	-3	-2	-1	0	1	2	3	4	
$x =$	3	$\frac{5}{4}$	0	$-\frac{3}{4}$	-1	$-\frac{3}{4}$	0	$\frac{5}{4}$	3	etc.
$y =$	-2	-1	0	1	2	3	4	5	6	

The curve is drawn in Fig. 89. It looks like a parabola. To determine

whether it is a parabola, eliminate the parameter t between the two given equations, thus obtaining the equation

$$x = \tfrac{1}{4}(y - 2)^2 - 1,$$

or

$$y^2 - 4y - 4x = 0.$$

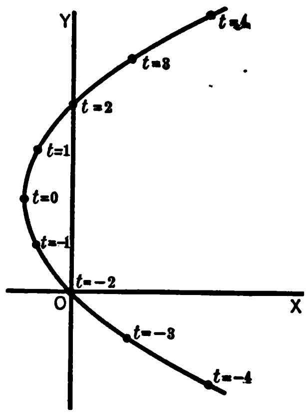


FIG. 89

which is the equation of a parabola.

3. Construct the locus whose parametric equations are $x = a\theta$, $y = b(1 - \cos\theta)$.

Assigning values to θ, and computing the corresponding values of x and y the results are tabulated as follows:

$\theta=0$	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	
$x=0$	$\frac{1}{6}\pi a$	$\frac{1}{3}\pi a$	$\frac{1}{2}\pi a$	$\frac{2}{3}\pi a$	$\frac{5}{6}\pi a$	πa	$\frac{7}{6}\pi a$	etc.
$y=0$	$b(1-\frac{1}{2}\sqrt{3})$	$\frac{1}{2}b$	b	$\frac{3}{2}b$	$b(1+\frac{1}{2}\sqrt{3})$	$2b$	$b(1+\frac{1}{2}\sqrt{3})$	

The curve is drawn in Fig. 90.

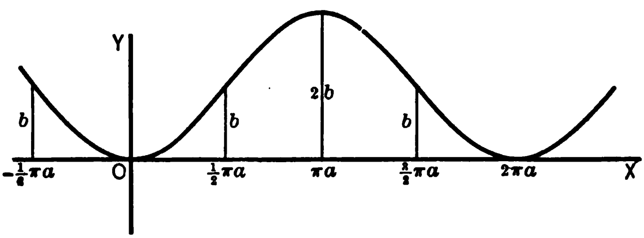


FIG. 90.

98. The cycloid.

DEFINITION. *If a circle rolls on a straight line the curve described by any fixed point on the circle is called a* **cycloid.**

Let r be the radius of the rolling circle, and OA, Fig. 91, the straight line on which it rolls. Let the point P of the circle describe the curve OHA, where O is the position of the describing

point where it touches the given line, and A the position it reaches after one complete revolution. The curve OHA is then a *cycloid,* sometimes called *one arch of the cycloid,* since as the circle continues to roll the point P will describe a succession of curves identical with OHA in form. The line OA is called the base of the cycloid.

As the circle rolls every point of it comes successively into contact with OA, hence the length of OA is the same as the

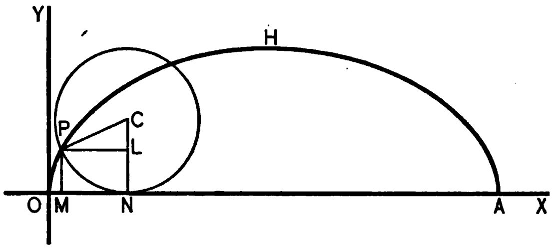


FIG. 91

circumference of the circle, $2\pi r$. For the same reason when the circle has turned through the angle $NCP = \theta$, as shown in the figure,

$$\textit{distance } ON = \textit{arc } PN = r\theta. \tag{i}$$

Take OA for X-axis, and O, the initial position of P for origin, then for the coordinates of P,

$$x = OM, \qquad y = MP. \tag{ii}$$

But

$$\begin{aligned} OM &= ON + NM = ON + LP \\ &= r\theta - r\sin\theta, \end{aligned} \tag{iii}$$

because by (*i*) $ON = r\theta$, and from the figure $LP = -r\sin\theta$. The negative sign is used for LP because it is measured in the negative direction for abscissas. Also,

$$MP = NL = NC + CL = r - r\cos\theta \tag{iv}$$

because

$$CL = -r \cos \theta.$$

The negative sign is used for CL because it is measured in the negative direction for ordinates. Hence from (*ii*), (*iii*) and (*iv*)

$$x = r(\theta - \sin \theta), \qquad y = r(1 - \cos \theta), \tag{10}$$

which are the parametric equations of the cycloid.

99. The epicycloid and hypocycloid.

I. THE EPICYCLOID.

DEFINITION. *If a circle roll on the outside of a fixed circle the curve described by any point of the rolling circle is called an* ***epicycloid***.

Let P be the generating point on the rolling circle, and A one of its positions where it touches the fixed circle. Take the center O of the fixed circle for origin, and OA for the X-axis. Let AQ be an arc of the epicycloid. The line OC, joining the centers of the two circles passes through K, their point of contact. Let $AOK = \theta$, $KCP = \phi$, $OK = R$, $CK = CP = r$. Draw NC, MP perpendicular to OX, and BCL parallel to OX, meeting MP produced at L. Angle $BCO = \theta$. Then for the coordinates of P,

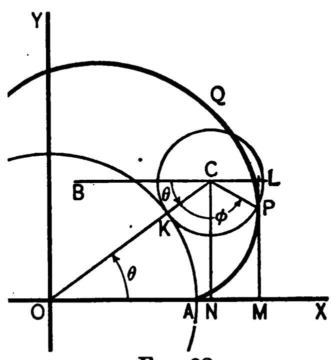


FIG. 92

$$x = OM = ON + NM = ON + CL, \tag{i}$$

where

$$ON = OC \cos NOC = (R + r) \cos \theta, \tag{ii}$$

and

$$CL = -CP \cos BCP = -r \cos (\phi + \theta). \tag{iii}$$

The negative sign is written in (*iii*) because CL is positive, while $\cos(\phi + \theta)$ is negative.

$$\therefore \quad x = (R + r)\cos\theta - r\cos(\phi + \theta). \qquad (iv)$$

Also,

$$y = MP = ML + LP = NC + LP, \qquad (v)$$

where

$$NC = OC \sin NOC = (R + r)\sin\theta, \qquad (vi)$$

and

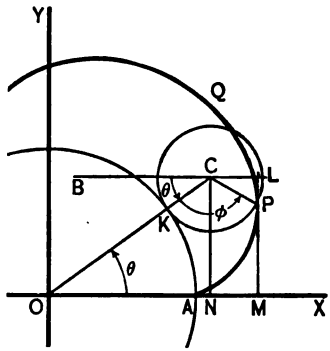


FIG. 92

$$LP = -CP\sin BCP$$

$$= -r\sin(\phi + \theta). \quad (vii)$$

The negative sign is written in (*vii*) because LP is negative, while $\sin(\phi + \theta)$ is positive.

$$\therefore \quad y = (R + r)\sin\theta - r\sin(\phi + \theta). \quad (viii)$$

To eliminate ϕ from equations (*iv*) and (*viii*) we have

$$\text{arc } KP = \text{arc } AK,$$

hence

$$r\phi = R\theta, \qquad \text{or} \qquad \phi = \frac{R}{r}\theta.$$

Substituting this value of ϕ in (*iv*) and (*viii*) the results are

$$\left.\begin{aligned} x &= (R + r)\cos\theta - r\cos\left(\frac{R + r}{r}\theta\right), \\ y &= (R + r)\sin\theta - r\sin\left(\frac{R + r}{r}\theta\right), \end{aligned}\right\} \qquad (11)$$

which are the parametric equations of the epicycloid.

II. THE HYPOCYCLOID.

DEFINITION. *If a circle roll on the inside of a fixed circle the curve described by any point on the rolling circle is called a* ***hypocycloid.***

Using the same notation as for the epicycloid the student will be able to show that the parametric equations of the hypocycloid are

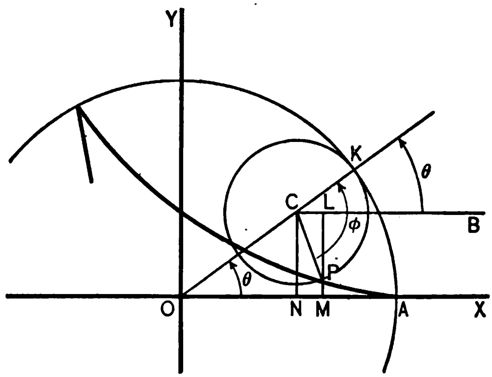


FIG. 93

$$\left.\begin{aligned} x &= (R - r)\cos\theta + r\cos\left(\frac{R - r}{r}\theta\right), \\ y &= (R - r)\sin\theta - r\sin\left(\frac{R - r}{r}\theta\right). \end{aligned}\right\} \qquad (12)$$

These equations may also be obtained by changing the sign of r in the equations of the epicycloid (11).

100. Special forms of the hypocycloid.

I. *When the diameter of the rolling circle is half the diameter of the fixed circle, the hypocycloid is a straight line.*

Making $R = 2r$ in equations (12) the results are

$$x = r\cos\theta + r\cos\theta = 2r\cos\theta, \qquad y = r\sin\theta - r\sin\theta = 0,$$

from the second of which it appears that the generating point travels on the X-axis, which proves the theorem.

II. *To find the equations of the hypocycloid when the diameter of the rolling circle is one fourth the diameter of the fixed circle.*

Making $R = 4r$ in (12) we have

$$x = 3r\cos\theta + r\cos 3\theta,$$
$$y = 3r\sin\theta - r\sin 3\theta,$$

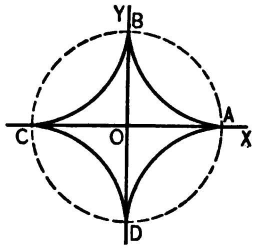


FIG. 94

or, substituting for $\cos 3\theta$ and $\sin 3\theta$ their values in terms of θ [see (28) p. x],

$$x = 4r\cos^3\theta, \quad y = 4r\sin^3\theta,$$

which are the required equations. Replacing $4r$ by its value $OA = R$, these equations become

$$\boldsymbol{x = R\cos^3\theta, \quad y = R\sin^3\theta.} \qquad (13)$$

The four points A, B, C, D in the figure are called **cusps**, and the curve is known as the **four-cusped hypocycloid.**

From (13) $x^{1/3} = R^{1/3}\cos\theta$, $y^{1/3} = R^{1/3}\sin\theta$, hence squaring and adding

$$\boldsymbol{x^{2/3} + y^{2/3} = R^{2/3}.} \qquad (14)$$

which is the equation in non-parametric form.

101. The involute of the circle.

DEFINITION. *Imagine a cord to be wound tightly around a circle, then as it is unwound, keeping it taut, the end of the cord describes the* ***involute of the circle.***

Let AQ be an arc of the involute generated by the point P, whose initial position was at A. LP is tangent to the circle at L. Take the center of the circle O for origin, and OA for the X-axis. Draw MP and NL perpendicular to OX, and KP parallel to OX. Let $AOL = \theta$ and $OL = r$. Since the triangles NOL, KLP are similar, the angle $KLP = \theta$. Then for the coordinates of the point P

$$x = OM = ON + NM = ON + KP, \qquad (i)$$
$$y = MP = NK = NL + LK, \qquad (ii)$$

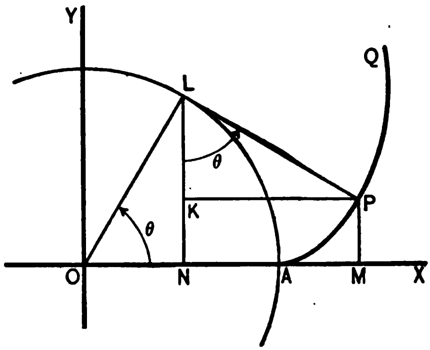


FIG. 95

where

$$ON = OL \cos\theta = r\cos\theta, \qquad NL = OL \sin\theta = r\sin\theta, \quad (iii)$$

and $KP = LP \sin\theta$, $LK = -LP\cos\theta$ (because LK is negative). But

$$LP = \text{arc } LA = r\theta,$$

hence

$$KP = r\theta \sin\theta, \qquad LK = -r\theta\cos\theta. \quad (iv)$$

Hence finally substituting from (*iii*) and (*iv*) in (*i*) and (*ii*),

$$\left.\begin{aligned} x &= r\cos\theta + r\theta\sin\theta, \\ y &= r\sin\theta - r\theta\cos\theta, \end{aligned}\right\} \quad (15)$$

which are the required parametric equations.

EXERCISES ON CHAPTER IX

1. Construct the loci determined by the following pairs of parametric equations, as in Exs. 2 and 3, Art. 97.

(*a*) $y = 4t$,	$x = 3t + 5$;	(*b*) $x = 8t$,	$y = 4t^2$;
(*c*) $x = 3\sin t$,	$y = 4\cos t$;	(*d*) $x = 4t$,	$y = -6t + 4$;
(*e*) $x = 4t^2$,	$y = 3t^3 - 1$;	(*f*) $y = 4\sec t$,	$x = 3\tan t$.

2. Determine the equation in x and y for each of the loci in Exercise 1, by eliminating the parameter between the given equations.

Ans. (*a*) $4x - 3y - 20 = 0$, (*b*) $16y = x^2$, (*c*) $16x^2 + 9y^2 = 144$.

3. Express the equation of the circle $x^2 + y^2 = r^2$ in parametric form.

4. Obtain parametric equations for the ellipse

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1.$$

5. When a circle rolls on a straight line the curve traced by a fixed point on a radius of the circle (or on the radius produced) is called a *trochoid*. If a is the radius of the circle, and b the distance of the tracing point from the center, show that the parametric equations of the trochoid are, with axes as in Art. 98,

$$x = a\theta - b \sin \theta, \qquad y = a - b \cos \theta.$$

6. Construct from their equations the trochoids (Ex. 5) for which $a = 2$, and $b = 1, \frac{3}{2}, 3$, respectively.

7. Construct the loci determined as follows:

(a) $x = 4(\theta - \cos \theta), \qquad y = 4(1 + \sin \theta).$
(b) $y = 2 \sin \theta - \sin 2\theta, \qquad x = 2 \cos \theta - \cos 2\theta.$
(c) $y = 3 \sin \theta - \sin 3\theta, \qquad x = 3 \cos \theta + \cos 3\theta.$
(d) $x = 2 \cos \theta + 2\theta \sin \theta, \qquad y = 2 \sin \theta - 2\theta \cos \theta.$

8. Find the equation in x and y for the locus in Ex. 7 (a) by eliminating θ. Ans. $x = 4 \sin^{-1} [\frac{1}{4}(y - 4)] - \sqrt{8y - y^2}$.

9. Obtain equations in parametric form for the parabola $y^2 = 2px$.

10. Prove that $x = a(\theta + \sin \theta)$, $y = -a(1 - \cos \theta)$ are the parametric equations of the cycloid when the origin is at the highest point on the curve, and the X-axis is the tangent at this point.

11. What does the trochoid (see Ex. 5) become when $b = a$? When $b = 0$? How do the trochoids in which $b < a$ differ from those in which $b > a$?

12. A straight line revolves about the point $(0, -b)$ intersecting the X-axis in the variable point M. On this line the point P is taken at the constant distance a from M. Prove that the parametric equations of the locus of P (called the *conchoid*) are

$$x = a \cos \theta + b \cot \theta, \qquad y = a \sin \theta.$$

CHAPTER X

POLAR COORDINATES

102. Polar coordinates of a point in a plane.

THEOREM. *The position of a point in a plane is determined by its distance and direction from a fixed point in the plane.*

Let A be a fixed point in the plane, and AB a fixed line drawn from A determining in the plane a fixed direction. Let P be any point in the plane. Then the position of P is determined if the length AP and the angle BAP are given.

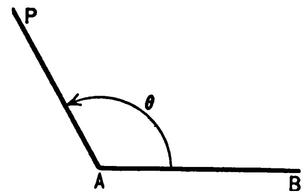


FIG. 96

The distance AP and the angle BAP are coordinates of the point P, since they locate the position of the point in the plane.

DEFINITIONS. Coordinates of the kind here described are called **polar coordinates.** The fixed point A is called the **pole** of the system, and the fixed line AB is called the **polar axis,** or the **initial line.** The coordinates AP and BAP of any point P are designated respectively by r and θ, and the point P is the point (r, θ). The distance r is called the **radius vector** of the point, and the angle θ is called the **vectorial angle.**

If in Fig. 96 $AP = r = 5$ and $BAP = \theta = 120°$, the point P is the point $(5, 120°)$ or $(5, \frac{2}{3}\pi)$.

103. Signs of the coordinates.—The sign of the vectorial angle θ is determined as in trigonometry. It is therefore positive when measured counter-clockwise from AB, and negative when measured in the opposite direction of rotation.

The sign of the radius vector r requires some explanation.

Every value of θ determines a line through A, as NM. One part of this line AM forms the terminal side of θ. The other portion, AN is the terminal side of θ produced through A. On every such line NM there is a positive and a negative direction. The positive direction is always from A on the terminal side of θ, and the opposite direction is negative. Thus in Fig. 97 for $\theta = BAM$, AM is the positive direction on the line NM, and AN the negative direction.

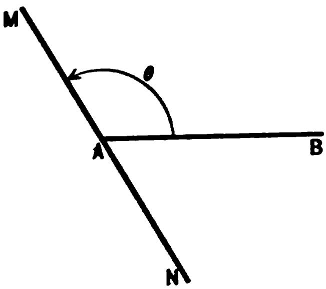


FIG. 97

A point P on AM, whose vectorial angle is $\theta = 120°$, Fig. 98, therefore has a positive radius vector. Let $AP = 3$ units, then P is the point $(3, 120°)$, or $(3, \frac{2}{3}\pi)$. The same value $\theta = 120°$ may also be used as the vectorial angle of a point Q on AN, but AQ the radius vector of Q being measured on the terminal side of θ produced through A, is negative. If the length of AQ is 3 units the coordinates of Q will be $(-3, 120°)$, or $(-3, \frac{2}{3}\pi)$.

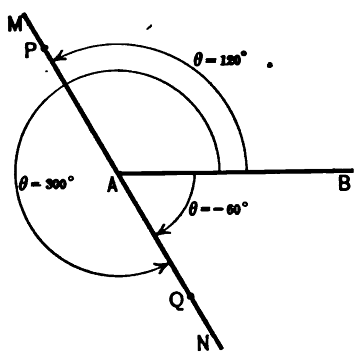


FIG. 98

Instead of taking $\theta = 120°$, the value $\theta = -60°$ may be used, since this also determines the line MN in the same position as before. In this case AN is the terminal side of θ. Hence AN is now the positive, and the negative direction, and the coordinates of P and Q

are $(-3, -60°)$, and $(3, -60°)$ respectively. Also since the angles $-60°$ and $+300°$ are the same, we may write $P = (-3, 300°)$, $Q = (3, 300°)$.

Any of the infinitely many ways whereby the angle which MN makes with AB can be expressed, may be used as the value of θ, and the radius vector of any point on MN will then be positive or negative according as the point lies on the terminal side of θ, or on the terminal side produced through A.

EXERCISES. 1. Locate each of the following sets of points, each set on one diagram

(*a*) $(2, 30°)$, $(4, 120°)$, $(2, \frac{1}{2}\pi)$, $(3, \frac{3}{2}\pi)$, $(4, \frac{7}{4}\pi)$, $(3, 281°)$, $(5, \pi)$, $(2, \frac{7}{120}\pi)$, $(3.7, 47° 18')$, $(1.47, 15° 14')$, $(0, 0)$, $(0, 4\pi)$.

(*b*) $(4, 75°)$, $(4, -75°)$, $(-4, 75°)$, $(-4, -75°)$.

(*c*) $(3, \frac{10}{3}\pi)$, $(3, -\frac{10}{3}\pi)$, $(-3, \frac{10}{3}\pi)$, $(-3, -\frac{10}{3}\pi)$.

(*d*) $(4, \frac{5}{6}\pi)$, $(4, -\frac{7}{6}\pi)$, $(-4, \frac{11}{6}\pi)$, $(-4, -\frac{1}{6}\pi)$, $(4, \frac{17}{6}\pi)$.

2. What is the distance between the points $(4, \frac{7}{6}\pi)$ and $(4, \frac{1}{6}\pi)$? between $(4, \frac{7}{6}\pi)$ and $(-4, \frac{7}{6}\pi)$? between $(4, \frac{1}{6}\pi)$ and $(-4, \frac{1}{6}\pi)$?

3. What angle with the polar axis is made by the line connecting $(4, \frac{1}{3}\pi)$ and $(4, -\frac{1}{3}\pi)$?

4. What is the direction of the line connecting $(6, \frac{1}{3}\pi)$ and $(6, \frac{2}{3}\pi)$?

5. On what curve are situated all points whose radius vector is 4? whose radius vector is a? What is the equation in polar coordinates of the circle whose radius is a, and whose center is at the pole?

6. What is the locus of all points whose vectorial angle is $\frac{1}{4}\pi$? whose vectorial angle is α? What is the equation in polar coordinates of the straight line which passes through the pole, making an angle α with the polar axis? Ans. $\theta = \alpha$.

104. Curve plotting in polar coordinates.

1. Plot the curve $r = 2a \cos \theta$.

By assigning values to θ in the equation, and computing the corresponding values of r, the coordinates of points on the curve are obtained and plotted on the figure. The curve is then traced through these points. This should be done in an

orderly manner beginning usually with $\theta = 0°$, and proceeding with values of θ at regular intervals. The curve proceeds from point to point in the same order as that in which the points are obtained as θ increases. The smaller the interval between successive values of θ the greater the number of points obtained, and hence the more accurately the curve can be traced. It is sometimes desirable to make this interval only 5° or 10°. Usually, however, careful examination of the equation will reveal how r varies as θ increases, and a smaller number of points will serve the purpose. In this example values of θ differing by 30° will be used. The following table shows how the work may be conveniently arranged.

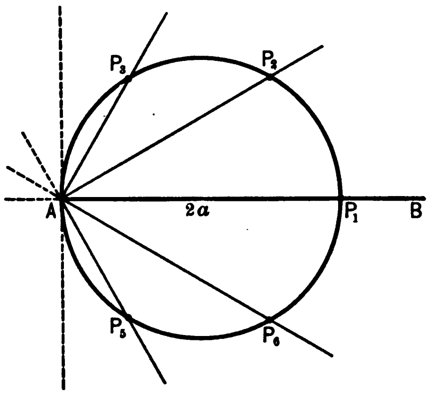


FIG. 99

θ	r	Point
0°	$2a$	P_1
30°	$1.73a$	P_2
60°	a	P_3
90°	0	A
120°	$-a$	P_5
150°	$-1.73a$	P_6
180°	$-2a$	P_1
210°	$-1.73a$	P_2
etc.	etc.	etc.

It should be noted that the point $(-2a, 180°)$ is the same as $(2a, 0°)$, and that as θ increases beyond 180° the series of points obtained is the same as that determined by the values of θ from 0° to 180°. This recurrence of points often happens in plotting equations containing trigonometric functions, and is a result of the periodicity of the values of these functions. The place in the series of values of θ where the recurrence begins

is not always 180°, but depends upon the form of the equation and upon the trigonometric functions involved in it. The curve in Fig. 99 is a circle. (See Art. 105).

2. Plot the curve $r = a \sin 3\theta$.

Taking values of θ at intervals of 10° the table of corresponding values of θ and r is as follows:

θ	r	θ	r
0°	0	100°	$-0.87a$
10°	$0.5a$	110°	$-0.5a$
20°	$0.87a$	120°	0
30°	a	130°	$0.5a$
40°	$0.87a$	140°	$0.87a$
50°	$0.5a$	150°	a
60°	0	160°	$0.87a$
70°	$-0.5a$	170°	$0.5a$
80°	$-0.87a$	180°	0
90°	$-a$	190°	$-0.5a$

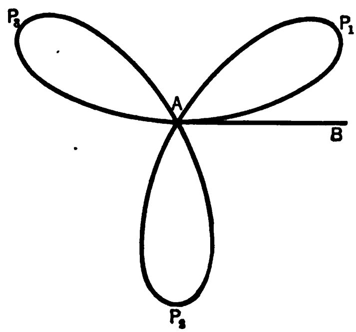


Fig. 100

The curve is completely traced as θ changes from 0° to 180°. The points P_1, P_2, P_3 are given respectively by the coordinates $(a, 30°)$, $(-a, 90°)$, $(a, 150°)$.

3. Plot the curve $r = 2 - \sin \frac{3}{2}\theta$.

Since the value of the sine of an angle is never greater than $+1$ nor less than -1, the values of r in this case are always positive and cannot be less than 1, nor greater than 3. Therefore the curve does not pass through the pole. The minimum value of r is $+1$, corresponding to values of θ for which

$$\sin \tfrac{3}{2}\theta = 1.$$

Similarly the maximum value of r is $+3$, corresponding to values of θ for which

$$\sin \tfrac{3}{2}\theta = -1.$$

The table of corresponding values of θ and r is now computed.

θ	r	Pt.	θ	r	Pt.	θ	r	Pt.	θ	r	Pt.
0°	2.0	1	210°	2.7	8	420°	3.0	15	630°	2.7	22
30°	1.3	2	240°	2.0	9	450°	2.7	16	660°	3.0	23
60°	1.0	3	270°	1.3	10	480°	2.0	17	690°	2.7	24
90°	1.3	4	300°	1.0	11	510°	1.3	18	720°	2.0	1
120°	2.0	5	330°	1.3	2	540°	1.0	19	750°	1.3	2
150°	2.7	6	360°	2.0	13	570°	1.3	20	etc.	etc.	etc.
180°	3.0	7	390°	2.7	14	600°	2.0	21			

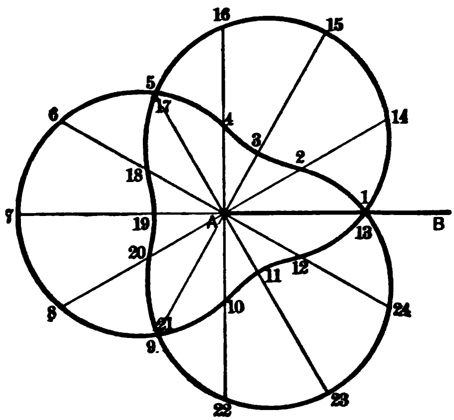


FIG. 101

In plotting this curve it must be noted that although when $\theta = 360°$, r has the same value as when $\theta = 0°$, the curve is not complete until $\theta = 720°$, after which the points recur in regular order.

4. Trace the curve $r = a\theta$.

This is one of a large class of curves called **spirals**. This is known as the *spiral of Archimedes*.

The values of θ are taken in radians. The values of r evidently increase indefinitely as θ increases.

The portion of the curve drawn solid is that given by positive

θ	r
0	0
$\frac{1}{4}\pi$	$0.79a$
$\frac{1}{2}\pi$	$1.57a$
$\frac{3}{4}\pi$	$2.36a$
π	$3.14a$
$\frac{5}{4}\pi$	$3.93a$

etc.

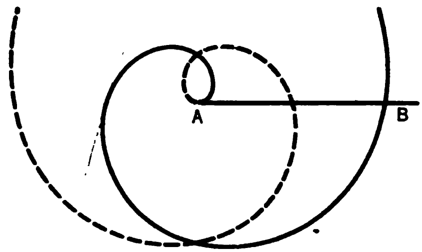


FIG. 102

values of θ, and the dotted portion is given by negative values of θ.

5. Plot the curve $r^2 = a^2 \cos 2\theta$.

In this equation if $\cos 2\theta$ is negative r is imaginary. Hence the curve does not exist for values of 2θ between 90° and 270°, that is for values of θ between 45° and 135°. For every value of θ which makes $\cos 2\theta$ positive r has two values numerically equal with opposite signs. Thus:

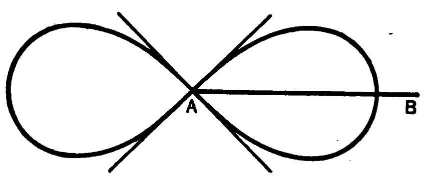


FIG. 103

θ	r
0°	$\pm a$
10°	$\pm 0.97a$
20°	$\pm 0.88a$
30°	$\pm 0.71a$
40°	$\pm 0.42a$
45°	0

Values of θ between 0° and − 45°, and between 135° and 225°, give the same series of values of r as those between 0° and 45°. This curve is called the *lemniscate of Bernouilli*.

EXERCISES. Plot the following curves in polar coordinates.

1. $r = 2a \sin \theta$.
2. $r = a \sin 2\theta$.
3. $r = a \csc 2\theta$.
4. $r\theta = a$.
5. $r = a \sin \frac{1}{2}\theta$.
6. $r = 2a \tan \theta$.
7. $r = a \cos 2\theta$.
8. $r = a \cos 3\theta$.
9. $r^2 = a^2 \cos \theta$.
10. $r = 10^\theta$.
11. $r = 2a \sec \theta$.
12. $r = a \tan 2\theta$.

13. $r = a(1 - \cos\theta)$.

14. $r = a(1 + \sin\frac{1}{2}\theta)$.

15. $r^2 = \dfrac{4}{2 - \cos\theta}$.

16. $r = \dfrac{4}{1 - 4\cos\theta}$.

17. $r = \dfrac{4}{1 - \cos\theta}$.

18. $r^2 = \dfrac{4}{1 - \cos\theta}$.

19. $r^2 = a^2(1 + \cos\theta)$.

20. $r = \dfrac{4}{1 - \cos 2\theta}$.

21. $r = a(1 + \frac{1}{2}\sin\theta)$.

22. $r = a(1 + 2\sin\theta)$.

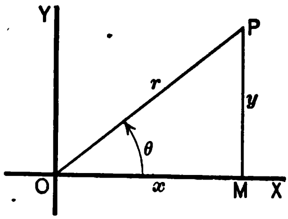


FIG. 104

105. Transformation of coordinates.—If the origin of a rectangular system of coordinates, and the positive extension of the X-axis be taken as the pole and polar axis, respectively, of a polar system, the following relations exist between the coordinates (x, y) and (r, θ) of any point P in the two systems.

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{1}$$

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}, \tag{2}$$

Only the positive value of $\sqrt{x^2 + y^2}$ is taken in (2) and in the following equations, because a positive value for r may always be taken. See Art. 103.

From (1) and the first equation in (2) we have also

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}, \tag{3}$$

which are often useful.

To illustrate the application of these formulas take the equation $y^2 = 2ax - x^2$ representing the circle with radius a, and center at $(a, 0)$. To find the polar equation of this circle write it in the form

$$x^2 + y^2 = 2ax,$$

and substitute $x^2 + y^2 = r^2$ from (2), and $x = r \cos \theta$ from (1). The result is $r^2 = 2ar \cos \theta$, or

$$r = 2a \cos \theta,$$

which is therefore the polar equation of the circle of radius a which passes through the pole, and has its center on the polar axis. See Fig. 99, p. 182.

As another example take the equation

$$r = a(\cos \theta - \sin \theta),$$

and transform it to rectangular coordinates. Using formulas (2) and (3) the result is

$$\sqrt{x^2 + y^2} = a\left(\frac{x}{\sqrt{x^2 + y^2}} - \frac{y}{\sqrt{x^2 + y^2}}\right)$$

or

$$x^2 + y^2 - a(x - y) = 0,$$

which is the equation of a circle passing through the origin, with its center at $(\frac{1}{2}a, -\frac{1}{2}a)$. See Art. 40.

106. Polar equation of the straight line.—Let HK be any straight line referred to the polar system in which A is the pole and AB the polar axis. Draw AM from the pole perpendicular to HK, and let $AM = p$, and the angle $BAM = \alpha$. These constants, p and α determine the position of the line, as in Art. 26. Let P be any point on the line with coordinates $AP = r$, $BAP = \theta$. Then in the right triangle AMP, the angle $MAP = \theta - \alpha$, and hence the required equation is

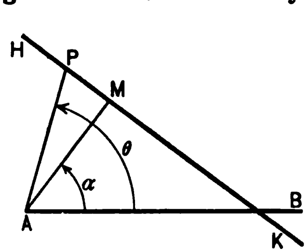


FIG. 105

$$r \cos (\theta - \alpha) = p. \tag{4}$$

107. Polar equation of the conics with focus at the pole. Let P be any point on a conic whose focus is at the pole A, and whose principal axis coincides with the polar axis AB. Let DN be the directrix. Then $DA = p$. From the definition of a conic, Art. 45,

$$AP = e \cdot NP, \qquad (i)$$

but $AP = r$, $NP = DM = DA + AM = p + r \cos \theta$. Hence substituting in (i)

$$r = e(p + r \cos \theta),$$

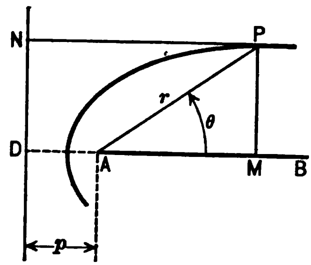


FIG. 106

from which

$$r = \frac{ep}{1 - e \cos \theta}. \qquad (5)$$

If the conic is a parabola $e = 1$, and equation (5) becomes

$$r = \frac{p}{1 - \cos \theta}. \qquad (6)$$

EXERCISES ON CHAPTER X

Normal Exercises

1. Locate on one diagram each of the following points: $(4, \frac{1}{3}\pi)$, $(5, 245°)$, $(-5, 245°)$, $(4, \frac{7}{3}\pi)$, $(4, \frac{4}{3}\pi)$, $(-4, \frac{4}{3}\pi)$, $(0, \frac{1}{2}\pi)$, $(4, \frac{1}{2}\pi)$.

2. Write four other pairs of polar coordinates each of which represents the same point as $(2, 60°)$.

3. Find the rectangular coordinates of the points of Ex. 1, the origin and X-axis coinciding with the pole and polar axis respectively.

4. Find the polar coordinates of the points whose rectangular coordinates are respectively $(-2, -2)$, $(4, 3)$, $(-3, 4)$, $(5, \frac{2}{3})$, $(0, 6)$, $(0, -6)$, $(6, 0)$, $(-6, 0)$, the pole and polar axis being as stated in Ex. 3.

5. Find the rectangular equation of the cardioid $r = a(1 + \cos \theta)$, the pole and polar axis being taken as stated in Ex. 3. Plot the curve from each equation on separate diagrams.

Ans. $a^2(x^2 + y^2) - (x^2 + y^2 - ax)^2 = 0.$

6. Find the polar equation of the semi-cubical parabola $ay^2 = x^3$, the pole and polar axis being taken as stated in Ex. 3. Plot the curve from each equation on separate diagrams. Ans. $r = a \tan^2 \theta \sec \theta$.

7. Plot the loci whose polar equations are

(a) $r = a \sin \frac{3}{2}\theta$, (b) $r = 1 + \cos \frac{2}{3}\theta$,

(c) $r = \dfrac{4}{4 - \cos^2 \theta}$, (d) $r = \dfrac{4}{2 - \cos \theta}$,

(e) $r^2 = \dfrac{432}{4 - \cos^2 \theta}$, (f) $r = \dfrac{6}{1 - 2 \cos \theta}$,

(g) $r \cos (\theta - 60°) = 2$, (h) $r^2 = \dfrac{144}{9 \sin^2 \theta - 16 \cos^2 \theta}$.

8. Find the polar equation of each of the following loci, and plot the graph.

(a) The lines whose inclination to the polar axis is 60°, and whose distance from the pole is 2.

(b) The parabola whose focus is one unit from the directrix.

(c) The ellipse in which $p = 2$ units, and $e = \frac{1}{2}$.

(d) The hyperbola in which $p = 2$ units, and $e = 3$.

General Exercises

9. Each of the equations which follow contains a general constant or parameter. Plot the locus of each equation on one diagram for several different values of the parameter which it contains, and note the differences in the resulting graphs due to the change in the value of the parameter.

(a) $r = a \cos \theta$, (b) $r = \cos m\theta$, (c) $r \cos \theta = a$,

(d) $r = 1 + \cos m\theta$, (e) $r = a + \cos \theta$,

(f) $r \cos (\theta - \alpha) = 4$, (g) $r(1 - \cos \theta) = p$,

10. Plot each of the following straight lines.

(a) $r \cos (\theta - \frac{1}{3}\pi) = 4$, (b) $r \cos (\theta + \frac{1}{2}\pi) = 2$,

(c) $r \cos (\theta - 327°) = 3$, (d) $r \cos \theta = 2$.

11. Transform each of the equations of Ex. 10 to rectangular coordinates with the origin at the pole and the X-axis coincident with the polar axis, and plot each line from its resulting equation.

Ans. (a) $x + y\sqrt{3} - 8 = 0$, (b) $y + 2 = 0$.

12. Find the equation in polar coordinates of the line which is

(*a*) perpendicular to the polar axis, and passes through $(4, 0)$.

Ans. $r \cos \theta = 4$.

(*b*) perpendicular to the line $\theta = \frac{1}{3}\pi$, and passes through $(4, 0)$.

Ans. $r \cos (\theta - 60^\circ) = 2$.

(*c*) parallel to the polar axis, and passes through $(6, \frac{1}{2}\pi)$.

13. Derive equation (4), p. 187, by transforming equation (9), p. 43, to polar coordinates.

14. Find the equations of the six conics determined by the following data, the focus in each case being at the pole, and the principal axis of the conic coincident with the polar axis.

(*a*) $p = 2, \quad e = 1$; (*b*) $p = 4, \quad e = \frac{1}{2}$; (*c*) $p = 3, \quad e = 2$;
(*d*) $p = 1, \quad e = \frac{1}{3}$; (*e*) $p = 4, \quad e = 1$; (*f*) $p = 3, \quad e = 3$.

15. Find the polar equation of the circle whose center is at $(4, 60^\circ)$, and radius 2. Ans. $r^2 - 8r \cos (\theta - 60^\circ) + 12 = 0$.

16. Show that $\theta^2 = \alpha^2$ represents two straight lines intersecting at the pole, and symmetrical to each other with respect to the polar axis. (See Ex. 6, p. 181.)

17. Develop a method for finding the distance between two points whose polar coordinates are given, and find the distances between the following pairs of points:

(*a*) $(4, \frac{1}{3}\pi), \quad (5, \frac{5}{6}\pi)$; (*b*) $(2, -\frac{1}{3}\pi), \quad (2, \pi)$;
(*c*) $(r_1, \theta_1), \quad (r_1, \theta_2)$; (*d*) $(4, \frac{1}{3}\pi), \quad (4, \frac{5}{6}\pi)$.

Ans. (*a*) $\sqrt{41}$, (*b*) $2\sqrt{3}$, (*c*) $2r_1 \sin \frac{1}{2}(\theta_1 - \theta_2)$.

18. Find the lengths of the semi-axes, the eccentricity, and the coordinates of center and foci of each of the following conics

(*a*) $r = \dfrac{4}{1 - 2\cos\theta}$, (*b*) $r = \dfrac{4}{2 - \cos\theta}$,

(*c*) $r = \dfrac{10}{2 - 5\cos\theta}$, (*d*) $r = \dfrac{4}{1 - 3\cos\theta}$.

Ans. (*a*) $a = \frac{4}{3}$, $b = \frac{4}{3}\sqrt{3}$, $e = 2$, $C = (\frac{8}{3}, \pi)$, foci $= (0, 0), (\frac{16}{3}, \pi)$.
(*b*) $a = \frac{8}{3}$, $b = \frac{4}{3}\sqrt{3}$, $e = \frac{1}{2}$, $C = (\frac{4}{3}, 0)$, foci $= (0, 0), (\frac{8}{3}, 0)$.

19. Determine the polar equation (*a*) of the ellipse, (*b*) of the hyperbola, the pole being at the center, and the polar axis coincident with the transverse axis.

CHAPTER XI

EMPIRICAL EQUATIONS

108. General remarks.—In the latter portion of Chapter II the problem of determining the equation of the locus of a point moving under prescribed conditions was briefly discussed, and some illustrations of the processes of solution were given. Elsewhere at numerous places throughout the preceding pages the same principles were employed, in deriving the equations of the straight line, circle, and conics, and of different loci connected with them.

A problem of somewhat different character will now be considered. The coordinates of a series of points are given. These will usually have been obtained from some practical investigation, as a result of some form of measurement, or from some statistical tabulation, as opposed to theoretical determination, and hence must be assumed to be subject to some degree of error, small in comparison with the given values themselves. It is required to find the equation of a curve which, as nearly as possible, passes through the given points. Sometimes in a problem of this kind it is known from theoretical considerations what is the general form of the equation of the curve on which the given points are supposed to lie, and sometimes nothing at all is known except the given coordinates. Two distinct varieties of the problem are therefore to be considered. In either case an equation of a curve derived from data of this kind is called an **empirical equation.**

109. Empirical equations, the law of the curve given. The method of procedure will be illustrated by examples.

1. The results of a test made upon a piece of steel at the

U. S. Arsenal, Watertown, Mass., to determine the elongation under different loads were as follows.

Load, lbs.	Elongation, inches	Load	Elongation	Load	Elongation
5,000	0.0013	35,000	0.0118	44,000	0.0148
10,000	0.0030	40,000	0.0135	45,000	0.0151
20,000	0.0060	41,000	0.0139	46,000	0.0156
30,000	0.0100	42,000	0.0142	47,000	0.0160
		43,000	0.0145		

It is required, using the corresponding pairs of numbers as coordinates of points to derive the equation of the locus on which the points lie.

It is known from practical and theoretical considerations that, within the elastic limit of the material, the elongation under different loads is proportional to the load. Hence the locus of these points is theoretically a straight line. Therefore in seeking its equation the proper form to use is

$$y = mx + b, \tag{i}$$

where m and b are to be determined so that the resulting line shall pass as near as possible to all of the given points.

Since the difference in the order of magnitude of these two sets of numbers is so great, it is obvious that if corresponding pairs were used as coordinates of points the resulting diagram would be of no practical value. Hence the scales used in the figure should be so adjusted that the numbers expressing abscissas and ordinates shall be of the same order of magnitude. This may be done by making the unit for the first column 1,000 lbs., and that for the second column 0.001 inch. Thus,

Load, unit = 1,000 lbs.	Elong., unit = 0.001 in.	Load	Elong.	Load	Elong.	Load	Elong.
5	1.3	35	11.8	42	14.2	45	15.1
10	3.0	40	13.5	43	14.5	46	15.6
20	6.6	41	13.9	44	14.8	47	16.0
30	10.0						

The next step is to plot the points whose abscissas and ordinates are the corresponding pairs of numbers in the first and second columns of this table, respectively. See Fig. 107.

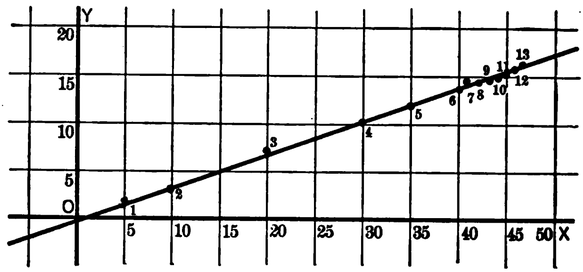


FIG. 107

Any two of the thirteen given points will determine a line which will approximate more or less closely to the required line. It is desirable to use points that do not lie close together, because if the points are close together a small error in the location of either of them will cause greater displacement of the line than would be produced by a similar error when the points are farther apart. Hence we choose the first and last points of the series, the points (5, 1.3), (47, 16). Substituting these values in (i) the results are

$$5m + b = 1.3,$$
$$47m + b = 16.0,$$

from which $m = 0.35$, and $b = -0.45$. Hence the equation of the line through these two points is

$$y = 0.35x - 0.45. \qquad (ii)$$

To test this equation compute the value of y for each given value of x, and compare the result with the corresponding given value of y. The differences are placed in the column Δ.

x	y Given	y Computed	Δ	x	y Given	y Computed	Δ
5	1.3	1.3	0	42	14.2	14.25	− 0.05
10	3.0	3.05	− 0.05	43	14.5	14.60	− 0.10
20	6.6	6.55	+ 0.05	44	14.8	14.95	− 0.15
30	10.0	10.05	− 0.05	45	15.1	15.30	− 0.20
35	11.8	11.80	0	46	15.6	15.65	− 0.05
40	13.5	13.55	− 0.05	47	16.0	16.0	0
41	13.9	13.90	0				

The table shows that all of the points except five lie below the line determined by the first and last. This result is not so satisfactory as it should be, and another line should be computed. For this purpose use the second point and the one next to the last, the points (10, 3) and (46, 15.6). Computing m and b as before the results are found to be $m = 0.35$, $b = -0.50$. The equation of this line is therefore

$$y = 0.35x - 0.50. \qquad (iii)$$

The student should compute a table of values for this equation similar to the one above. It will be found that the line (*iii*), determined by the second and twelfth points, passes through the fourth, sixth and eighth also, while of the rest of the points five lie above the line and three below it. It may be noted also that the sum of the positive Δ's thus obtained is numerically equal to the sum of the negative Δ's. Equation (*iii*) may therefore be accepted as a satisfactory result.

A systematic method for handling data of this kind for the purpose of arriving at the most probable result is based upon what is known as the method of least squares. It is, however, beyond the scope of this book.

2. When a gas is compressed, or allowed to expand, under such circumstances that heat is neither gained nor lost during the process, the gas is said to be compressed or to expand adiabatically. The equation in this case connecting the pressure

and volume of the gas is

$$pv^n = k, \tag{i}$$

where p = pressure, v = volume, and n and k are constants.

A certain body of air was compressed adiabatically, and corresponding pressures and volumes were observed to be as follows:

p = 1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
v = 8.78	7.17	6.12	5.38	4.82	4.38	4.03	3.74

Determine the values of n and k which will give the curve passing as nearly as possible through the eight points which have these pairs of values of p and v as coordinates.

First plot the points using the value of p as abscissas, and the corresponding values of v as ordinates. The first attempt to reach a satisfactory solution may be made by finding the curve whose equation has the form (i) which passes through the first and last points of the series, the points (1.2, 8.78) and (4.0, 3.74). Substituting these pairs of values in (i) the two resulting equations are

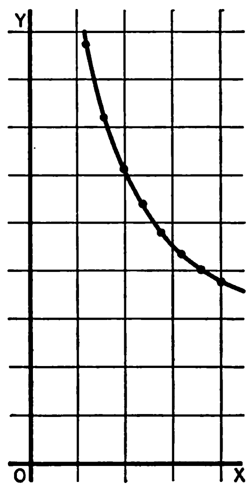


FIG. 108

$$1.2(8.78)^n = k, \qquad 4.0(3.74)^n = k, \tag{ii}$$

which are to be solved for n and k. Dividing the first by the second the result is

$$\left(\frac{8.78}{3.74}\right)^n = \frac{4.0}{1.2},$$

and taking the logarithms of both sides

$$n(\log 8.78 - \log 3.74) = \log 4.0 - \log 1.2$$

or

$$0.3706n = 0.5229,$$

from which

$$n = 1.411.$$

Substituting this value of n in the first equation of (*ii*) the value of k is

$$k = 1.2(8.78)^{1.411},$$

or

$$\log k = \log (1.2) + 1.411 \log (8.78)$$

$$\therefore \quad \log k = 1.4103, \text{ from which } k = 25.73.$$

Hence the curve of the required type which passes through the first and last points of the series has the equation

$$pv^{1.411} = 25.73. \qquad (iii)$$

To test this result substitute in it the other six given values of p and compare the resulting values of v computed from the equation with the corresponding given values. The results appear in the following table

p =	1.6	2.0	2.4	2.8	3.2	3.6
v (given) =	7.17	6.12	5.38	4.82	4.38	4.03
v (computed) =	7.16	6.11	5.37	4.82	4.38	4.03

The computed values of v agree so closely with the given values that equation (*iii*) may be accepted as a satisfactory result. Had it not been so our next attempt would have been to find the curve which passes through the first and seventh points, then through the second and eighth. If the results obtained from these two curves should still be unsatisfactory, the means of the values of n and k from the last two determinations could be used, or the means taken from all three determinations.

110. Empirical equations, the type of curve not given. This problem arises when a series of corresponding pairs of

numbers is given, which can be used as coordinates of points, but where the law connecting them either is unknown, or is not known with sufficient definiteness to be expressed by an equation. For example, the speed of a motor boat is a function (amongst other things) of the rate of consumption of gasoline. The hourly rate of consumption can be taken as abscissas and the resulting speeds as ordinates, and a curve drawn which will express graphically this relation.

The procedure in a case of this kind is first to plot the given points on a figure, adjusting the scales used, if necessary, so that the numbers expressing the abscissas and ordinates shall be of the same order of magnitude. The general shape of the curve can then be seen, and a form of equation adopted for trial which corresponds to the observed form of curve. The following examples illustrate the process.

1. Given the points whose coordinates are

$x = 0$	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$y = 8.10$	9.48	11.65	14.42	17.70	21.64	26.12	31.22	37.05

Find a good approximation to the equation of the curve on which they lie.

The points are plotted in Fig. 109, the unit for x being ten times the unit for y. A convenient form of equation to use for a case of this kind, where in general y increases as x increases, is

$$y = a + bx + cx^2 + dx^3 + \cdots. \qquad (i)$$

The solution of the problem consists in finding the values of a, b, c, etc., in (i), which will give the equation of a curve passing through a selected number of the given points, and as nearly as possible through the rest. An equation of the form (i) is chosen containing a definite number of terms on the right, and hence a definite number of undetermined coefficients, a, b, c, etc. An equal number of the given points must then be chosen,

so that when their coordinates have been substituted for x and y there will be as many equations in a, b, c, etc., as there are quantities to be determined. In general the greater the number of terms taken in (*i*) the more accurate the result, but on the

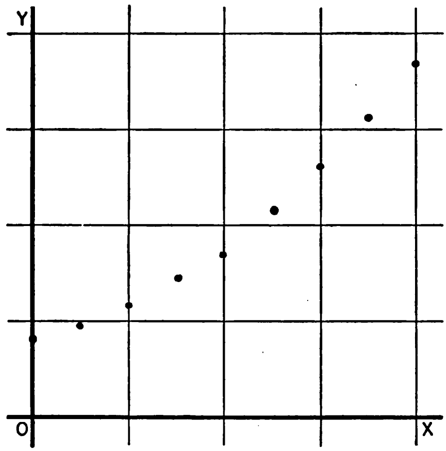


FIG. 109

other hand the labor of computing the undetermined coefficients a, b, c, etc., increases rapidly with their number, so that practically it is desirable to use an equation with as few terms as will give a satisfactory result. We begin therefore with

$$y = a + bx + cx^2, \qquad (ii)$$

the curve corresponding to which can be made to pass through any three points of the series, because the equation contains three undetermined coefficients, a, b, c.

Taking the two end points of the series and the middle point (0, 8.10), (2.0, 17.70), (4.0, 37,05) and substituting in (*ii*) we have

$$\begin{aligned} a &= 8.10, \\ a + 2b + 4c &= 17.70, \\ a + 4b + 16c &= 37.05, \end{aligned}$$

from which $a = 8.10$, $b = 2.362$, $c = 1.2188$. Hence (*ii*) becomes

$$y = 8.10 + 2.362x + 1.2188x^2. \qquad (iii)$$

If the ordinates to this curve (*iii*) be computed for the other given values of x they will be found to be

$x = 0.5$	1.0	1.5	2.5	3.0	3.5
$y = 9.59$	11.68	14.39	21.62	26.16	31.30

Comparing these with the corresponding given values of y it will be seen that the differences are all small. The curve corresponding to (*iii*), therefore, represents fairly well the relation connecting the given pairs of values of x and y.

Equation (*iii*) may therefore be used for computing values of y in this series corresponding to values of x lying between 0 and 4. An equation derived in this way should never be used, however, to extend the series in either direction beyond the limits fixed by the given values. Beyond these limits the curve corresponding to the assumed equation, and the curve which would correspond to the extended series may differ very widely.

If it should be impossible to reach a satisfactory result by the use of equation (*ii*), an additional term from (*i*) should be added and the process repeated with the extended equation.

2. Find the equation of a curve which as nearly as possible shall pass through the points

$x = 1$	2	3	4	5	6	7	8
$y = 14$	12.1	11.4	11.1	10.8	10.7	10.6	10.5

These points are plotted in Fig. 110, where one unit for x is the same as two units for y.

Since the value of y tends to decrease as x increases, the equation

$$y = a + \frac{b}{x} \qquad (i)$$

may be used. This equation contains two undetermined coefficients a and b, and therefore a curve with equation of this form can be determined so as to pass through any two of the eight given points. Using the first and last points of the series,

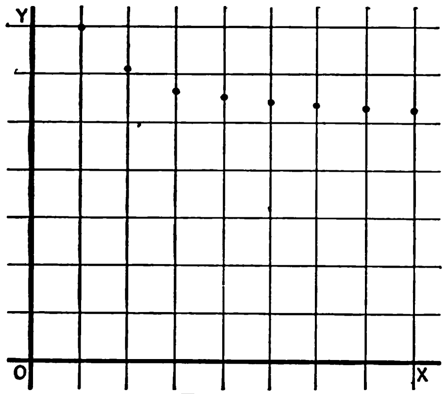


FIG. 110

and substituting in (i), the results are

$$a + b = 14,$$
$$a + \tfrac{1}{8}b = 10.5,$$

$\therefore \tfrac{7}{8}b = 3.5$, or $b = 4$, and hence $a = 10$. The resulting equation is therefore

$$y = 10 + \frac{4}{x}. \qquad (ii)$$

Substituting in (ii) the other six given values of x, the resulting values of y are

$x = 2$	3	4	5	6	7
$y = 12$	11.3	11.0	10.8	10.7	10.6

which agree closely with the given values of y.

111. Concluding remarks.—It is easily seen that each problem of the kind discussed in this chapter stands by itself.

Only broad general principles of procedure can be prescribed, as has been indicated in the examples given, and the details in each case must be adapted to the special requirements which it presents. The aim should be to produce an equation which expresses the law of variation of the data throughout the range for which the data are given, and within their limit of error. It may differ widely in form from the true theoretical function, and for this reason, in using it to obtain values in addition to those given it should never be used beyond the limits within which the data lie, without careful investigation as to its reliability outside of these limits. As a rule it can safely be assumed to coincide closely with the true function only within this range.

For example an equation for the curve traced through the points

$x = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$y = 0.100$	0.199	0.295	0.389	0.479	0.565	0.643	0.716

would be

$$y = x - \tfrac{1}{6}x^3 \qquad (i)$$

which agrees closely with these values, and this equation could very properly be used to express the law connecting them, within the limits here given. The equation

$$y = \sin x, \qquad (ii)$$

however, is also satisfied by the same numbers. Now if it be desired to extend the series of numbers beyond these limits it is found that

when $x = 1.0$	1.2	1.5	2.0	etc.
y, from $(i) = 0.833$	0.912	0.937	0.667	
y, from $(ii) = 0.842$	0.932	0.997	0.909	

so that the divergence of the two series of values of y increases rapidly. Hence, unless there is some means of determining which of the two equations, (i) or (ii) is the expression of the law upon which the numbers are based, the values thus derived,

by extending the series beyond the range of the data, are of no value.

If the results of a first try at a solution do not turn out satisfactorily another effort must be made either by extending the equation used in the first attempt by the addition of one or more terms, or by the use of some other form of equation appropriate to the arrangement of the given points.

In addition to equations of the forms used in examples 1 and 2, Art. 110, others which will sometimes be found useful are

$$y = cx^n,$$

where c and n are undetermined quantities;

$$\log y = a + bx,$$
$$\log y = cx^n,$$
$$y = a + \frac{b}{x^2},$$

where a, b, c and n are undetermined quantities.

Other forms may also be found amongst the curves given for reference in the Appendix.

EXERCISES ON CHAPTER XI

In the following exercises the units for abscissas and ordinates should be so chosen that the resulting numbers used as the coordinates of points shall be of the same order of magnitude. See Ex. 1, p. 191.

1. In each of the following cases find an equation of the form (i), p. 197, which closely represents the data.

(a) *Tensile Tests of a Steel Wire*

Load	Elongation	Load	Elongation	Load	Elongation
5,000	0.00000	25,000	0.00073	50,000	0.00170
10,000	0.00019	30,000	0.00090	55,000	0.00190
15,000	0.00037	40,000	0.00130	60,000	0.00210
20,000	0.00055	45,000	0.00150	65,000	0.00232

(*b*) *Tensile Test of a Bronze Specimen*

Load	Elongation	Load	Elongation	Load	Elongation
4,000	0.00013	12,000	0.00076	18,000	0.00158
6,000	0.00017	14,000	0.00098	20,000	0.00198
8,000	0.00042	16,000	0.00124	22,000	0.00243
10,000	0.00058				

(*c*) *Temperature Corrections for a Pyrometer*

T = temperature, C = correction.

$T = 800$	900	1,000	1,100	1,200	1,300	1,400	1,500
$C = 95$	85	75	65	55	45	35	25

(*d*) *Temperature Corrections for a Pyrometer*

T = temperature, C = correction.

$T = 700$	800	900	1,000	1,100	1,200	1,300	1,400
$C = 1$	2	4	7	10	15	20	26

(*e*) *Candle Power of Some Tungsten Lamps*

V = voltage, C = candle power.

$V = 65.0$	70.7	75.0	80.0	86.3	90.0	100.0	105.0	108.0	112.0
$C = 6.66$	9.30	11.71	15.03	20.06	23.49	34.67	41.40	45.82	52.21

2. To determine the tensile strength of three lots of steel specimens were taken and tested with results as given below. Find for each specimen the equation of the straight line which best represents the data.

Load, Pounds	Elongation in Inches		
	Specimen *a*	Specimen *b*	Specimen *c*
5,000	0.0006	0.0011	0.0015
10,000	0.0010	0.0024	0.0031
15,000	0.0016	0.0039	0.0048
20,000	0.0020	0.0050	0.0064
25,000	.....	0.0060	0.0079
30,000	0.0030	0.0072	0.0097
35,000	.....	0.0087	0.0111
40,000	0.0042	0.0098	
45,000	0.0047	0.0109	
50,000	0.0050		
55,000	0.0053		
60,000	0.0059		

3. Find the equation of the form (*i*), p. 197, which best fits each of the following sets of points. The values of x are the same for each set of points.

$x = 0$	1	2	3	4	5	6	7	8
$y_1 = 3.2$	2.5	4.4	7.1	16.6	26.8	39.7	55.4	73.8
$y_2 = 3.1$	7.7	12.2	16.4	19.3	20.7	18.8	13.3	4.1
$y_3 = 4.12$	6.33	8.57	10.79	13.01	15.23	17.45	19.71	21.95
$y_4 = 1.3$	5.7	9.6	12.4	14.3	14.7	14.0	13.3	10.7

4. Determine the equation of the form

$$I = \frac{ae^{bd}}{d},$$

which best represents the following data obtained from tests of wireless current received at varying distances from a constant source. d = distance, I = current, e = Napierian base.

$d = 300$	400	500	600	800	1,000	1,200	1,500	2,000
$I = 410$	300	225	160	95	59	34	19	5

5. For each of the sets of points below find a good approximation to the equation of the curve passing through them. In each case the proper form of equation to use is one of those given in Art. 111.

(*a*)	$x = 1.0$	1.2	1.4	1.6	1.8	2.0
	$y = 4.20$	5.36	6.58	7.86	9.19	10.58
(*b*)	$x = 0.5$	1.0	1.5	2.0	2.5	3.0
	$y = 13.9$	5.0	3.3	2.8	2.5	2.3
(*c*)	$x = 1.2$	1.6	2.0	2.4	2.8	3.2
	$y = 7.33$	9.71	12.81	16.88	22.27	29.44

(*d*)	$x = 0.2$	0.4	0.6	0.8	1.0	1.2	1.4
	$y = 0.96$	0.86	0.69	0.52	0.38	0.24	0.14

6. For each of the sets of points below determine a good approximation to the equation of the curve which passes through them. The values of x are the same for each set of points.

$x = 0.5$	1.0	1.5	2.0	2.5	3.0
$y_1 = 4.72$	3.10	2.57	2.28	2.13	2.04
$y_2 = 0.33$	2.49	8.42	20.12	39.0	67.6
$y_3 = 1.78$	2.00	1.94	1.62	0.94	0.01
$y_4 = 0.83$	1.14	1.58	2.31	3.48	5.38

CHAPTER XII

EXTENSION OF COORDINATE GEOMETRY TO SOME SPACE PROBLEMS

112. Rectangular coordinates in space.

THEOREM. *The position of a point in space is determined when its distances from three planes, mutually perpendicular to each other are known.*

Let XOY, YOZ, ZOX be three mutually perpendicular planes, and P any point in space. From P draw a perpendicular to the plane XOY, intersecting this plane at M. The position of M in the plane XOY is determined by the coordinates $ON = x$, $NM = y$. Hence these two distances together with the distance $MP = z$ determine the position of P in space.

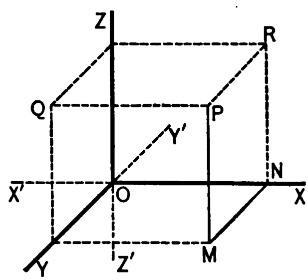


FIG. 111

Complete the rectangular parallelopiped which has O and P for opposite vertices, and ON, NM, MP for its three dimensions. The edges $QP = ON = x$, and $RP = NM = y$ are respectively the distances of P from the planes YOZ, ZOX. Hence the three distances QP, RP, MP determine the position of P.

DEFINITIONS. I. *The Coordinate System.* The three planes, XOY, YOZ, ZOX, are called the **coordinate planes,** their common point of intersection, the **origin,** and the three lines $X'X$, $Y'Y$, $Z'Z$, in which they intersect, the **coordinate axes.** The distances of a point from the three coordinate planes are called

the **coordinates** of the point, and are represented by the letters x, y, z, as indicated above, the x of a point being its distance from the plane YOZ, measured parallel to $X'X$, etc.

II. *Signs of the coordinates.* Each coordinate of a point is measured *from* the corresponding coordinate plane, and hence is positive or negative according as it is measured in one direction or the opposite. As the figure is ordinarily drawn (see Fig. 111), the positive directions for x, y, z are respectively OX, OY, OZ. The three coordinates of a point are all positive in one, and only one, of the eight solid angles (*octants*) formed by the intersection of the three coordinate planes.

III. *Notation.* In writing the coordinates of a point they are enclosed in parenthesis, and written in the order x, y, z. Thus the point $(3, -2, 4)$ is the point for which $x = 3$, $y = -2$, $z = 4$. This point is found in the octant $O - XY'Z$, Fig. 111, since its y is negative, while its x and z are positive.

IV. *Drawing the Figure.* In drawing figures for geometry of space a conventional perspective is used. See Fig 111. The axes $X'X$, $Z'Z$ lie in the plane of the paper and are drawn at right angles. The axis $Y'Y$, which is perpendicular to both of the others, is drawn so as to make apparently equal angles with OX', OZ'. All parts of the figure which lie in the plane XOZ, or in planes parallel to XOZ, are drawn in their true shape, but when they lie in other planes they are more or less distorted. Distances parallel to OX or OZ are drawn in their true length, but those parallel to OY must be foreshortened.

113. The distance between two points.—Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any two points in space. Draw P_1M_1 and P_2M_2 perpendicular to the plane XOY. In the plane $P_1M_1M_2P_2$ draw P_2Q parallel to M_2M_1, meeting M_1P_1 in Q. The triangle P_1QP_2 is right-angled at Q, hence

$$P_1P_2 = \sqrt{\overline{P_2Q}^2 + \overline{QP_1}^2},$$

where $P_2Q = M_2M_1 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, by (3), p. 6,

and $QP_1 = M_1P_1 - M_1Q = M_1P_1 - M_2P_2 = z_1 - z_2.$

$$\therefore \quad P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}. \tag{1}$$

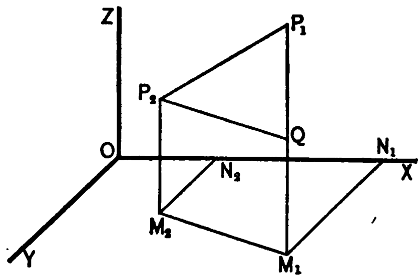


FIG. 112

EXERCISES. 1. Draw a set of coordinate planes, and using any convenient unit of length, locate the points (3, 2, 4), (4, 3, 2), (− 3, 2, − 2), (0, 4, − 3), (0, 0, 5), (2, − 3, 5), (− 2, − 3, − 4).

2. What are the coordinates of a point on the X-axis at the distance 4 from the origin?

3. A point is on the Z-axis at the distance − 4 from the origin, what are its coordinates?

4. What is the locus of all points for which (a) z is zero, (b) x is zero, (c) x and y are both zero, (d) x is + 2, (e) x is 2 and y is 3?

5. What is the distance from the origin to each of the points (x_1, y_1, z_1), $(-x_1, -y_1, z_1)$, $(x_1, -y_1, z_1)$, $(x_1, y_1, -z_1)$, $(x_1, -y_1, -z_1)$, $(-x_1, -y_1, -z_1)$? Which of these points are so situated that the straight line joining them passes through the origin? Which of them are symmetrically situated with respect to a coordinate plane? with respect to a coordinate axis?

6. Determine the lengths of the sides of the triangle whose vertices are the points (2, − 4, 7), (3, − 2, 0), (4, − 5, 4).

Ans. $3\sqrt{6}$, $\sqrt{14}$, $\sqrt{26}$.

7. Find the lengths of the sides and diagonals of the quadrilateral whose vertices in order are the points (3, 2, 4), (2, − 3, 5), (1, − 2, − 3), (3, 2, − 1). Ans. Sides, $3\sqrt{3}$, $\sqrt{66}$, $2\sqrt{6}$, 5; Diagonals $\sqrt{69}$, $\sqrt{62}$.

114. Projections.—I. The **orthogonal projection** of a point on a line or plane is the foot of the perpendicular from the point to the line or plane. Let P be any point, RS any plane, and PM the perpendicular from P to the plane, then M is the

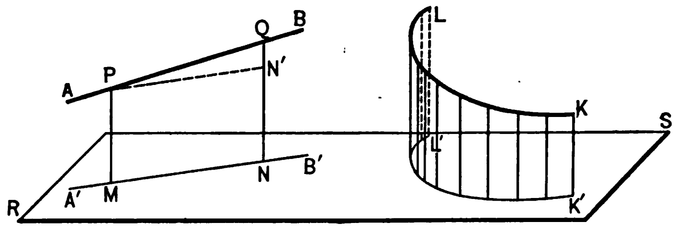


FIG. 113

orthogonal projection of P on the plane RS. M is also the orthogonal projection of P on the line $A'B'$ in the plane RS, passing through M.

When the term projection is used in the following pages orthogonal projection will always be meant.

II. The line or plane on which a projection falls is called the **line** or **plane** of **projection.**

III. The projection of a line or curve on a plane is the locus of the projections on the plane of all the points on the line or curve. As proved in elementary geometry the projection of a straight line on a plane is in general a straight line. In Fig. 113 $A'B'$ is the projection of AB on the plane RS.

IV. The projection of a curve on a plane is in general another curve. Thus $L'K'$ is the projection of LK. In particular, if the curve projected lies in a plane parallel to the plane of projection, the projection is identical in shape and size with the curve projected.

115. Direction angles and cosines.

DEFINITION. *The angles which a straight line makes with the three coordinate axes are called the* ***direction angles*** *of the line,*

and the cosines of these angles are called the **direction cosines** *of the line.*

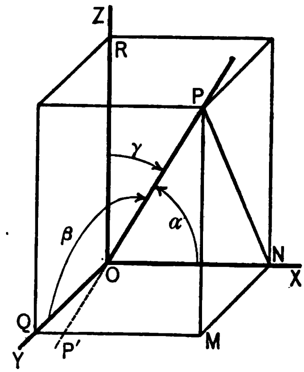


FIG. 114

If the line pass through the origin, as OP, Fig. 114, the direction angles are $XOP = \alpha$, $YOP = \beta$, $ZOP = \gamma$.

Since the angle between two non-intersecting lines in space is defined to be the angle formed by two lines drawn through a common point parallel respectively to the non-intersecting lines, the direction angles of a line which does not pass through the origin are the same as those of a parallel line through the origin.

NOTE. The direction angles of a line $P'P$ passing through the origin may be taken as the angles which either OP, or OP' makes with the positive directions on the three axes. Either of these sets of angles determines completely the direction of the line in space.

THEOREM I. *The sum of the squares of the direction cosines of any line is equal to unity.*

From what has just been said it will be sufficient to prove the theorem for a line passing through the origin.

Let OP, Fig. 114 be the line. From P let fall a perpendicular to OX. This will meet OX at N, where the perpendicular from M meets OX. The triangle OPN is right angled at N. Let $OP = r$. Therefore

$$\cos\alpha = \frac{ON}{OP} = \frac{x}{r}. \qquad (i)$$

Similarly

$$\cos\beta = \frac{OQ}{OP} = \frac{y}{r}, \qquad \cos\gamma = \frac{OR}{OP} = \frac{z}{r}. \qquad (ii)$$

Squaring and adding,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{x^2 + y^2 + z^2}{r^2}. \qquad (iii)$$

But OP is a diagonal of the rectangular parallelopiped whose three conterminous edges are $ON = x$, $OQ = NM = y$, $OR = MP = z$. Hence $x^2 + y^2 + z^2 = r^2$.

$\therefore$ from (iii)

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \qquad (2)$$

DEFINITION. The distance r of a point from the origin is called the **radius vector** of the point.

From (i) and (ii) above

$$x = r \cos \alpha, \qquad y = r \cos \beta, \qquad z = r \cos \gamma. \qquad (3)$$

THEOREM II. *The direction cosines of the line joining two given points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ are*

$$\cos \alpha = \frac{x_2 - x_1}{d}, \qquad \cos \beta = \frac{y_2 - y_1}{d}, \qquad \cos \gamma = \frac{z_2 - z_1}{d}, \qquad (4)$$

where d is the distance P_1P_2, Fig. 115.

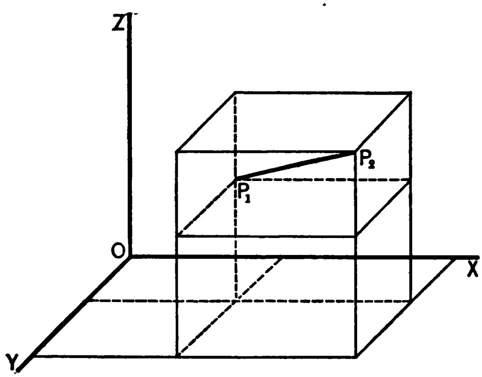


FIG. 115

The proof is left to the student.

116. Cylindrical Coordinates.—The position of a point in space is determined by its distance from a fixed plane and the polar coordinates in this plane of the projection of the point upon it. These coordinates are called **cylindrical coordinates.**

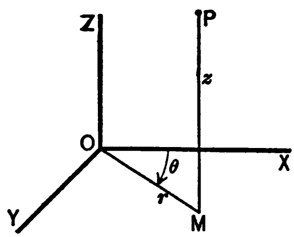


FIG. 116

Let P be the point, XOY the fixed plane, and OX the polar axis in this plane. Then the cylindrical coordinates of P are

$$MP = z, \qquad OM = r, \qquad XOM = \theta. \tag{5}$$

NOTE. The letter r is used here to represent OM, instead of OP as in Arts. 115, 117.

117. Spherical coordinates.—Let ZOX be a fixed plane, and XOY, YOZ two planes perpendicular to XOZ and to each other. Let P be any point in space. Draw OP, let fall PM perpendicular to XOY, and draw OM. Then

$$OP = r, \qquad XOM = \theta, \qquad ZOP = \gamma \tag{6}$$

are the **spherical coordinates** of P.

These coordinates determine the position of P because θ determines the plane through OZ which contains P, and (r, γ) are the polar coordinates of P in this plane with reference to OZ as polar axis.

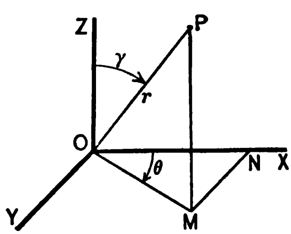


FIG. 117

The rectangular coordinates of P may be expressed in terms of r, θ, γ, as follows. Draw MN perpendicular to OX, then

$$x = ON = OM \cos\theta, \qquad y = NM = OM \sin\theta. \tag{i}$$

In the right triangle OMP, right-angled at M, the angle $MPO = \gamma$, hence

$$OM = r \sin \gamma, \qquad \text{and} \qquad z = MP = r \cos \gamma. \qquad (ii)$$

Substituting the value of OM from (ii) in (i) the results are

$$\boldsymbol{x = r \cos \theta \sin \gamma, \qquad y = r \sin \theta \sin \gamma, \qquad z = r \cos \gamma.} \qquad (7)$$

Conversely from equations (7), or from the figure

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1} \frac{y}{x}, \\ \gamma &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}. \end{aligned} \qquad (8)$$

EXERCISES. 1. Find the direction cosines of the lines determined by the following pairs of points, taking the direction for each line from the first to the second given point.

(a) (0, 0, 0), (1, 3, 7);
(b) (1, − 4, 3), (3, 2, 5);
(c) (− 1, 2, − 4), (2, − 7, 3);
(d) (0, 5, 0), (5, 0, 0);
(e) (0, 0, 4), (5, 4, 0);
(f) (1, 2, 3), (− 1, − 2, − 3).

Ans. (a) $\frac{1}{59}\sqrt{59}, \frac{3}{59}\sqrt{59}, \frac{7}{59}\sqrt{59}$.

2. What are the direction cosines (a) of the positive direction of the X-axis, (b) of the negative direction of the Z-axis?

3. What are the direction cosines of the three lines joining the origin to the three points (1, 1, 0), (1, 1, 1), (− 1, − 1, − 1)?

4. Find the lengths of the sides and diagonals of the quadrilateral whose vertices in order are the points (1, 2, 4), (3, 2, − 1), (7, 5, − 3), (5, 5, 2). Prove that the figure is a parallelogram, by showing that the opposite sides are parallel.

Ans. Sides all equal $\sqrt{29}$; diagonals, $\sqrt{94}$, $\sqrt{22}$.

5. Find the lengths of the projections of the line from (2, − 3, 3) to (− 2, 2, 4) on the X-, Y-, and Z-axes respectively.

Ans. − 4, 5, 1.

6. Show that (2, − 3, 1) is the center of a sphere which passes through the points (8, 0, 3), (4, 3, 4), (0, − 9, − 2), (5, − 1, 7), (9, − 3, 1).

7. Two of the direction cosines of a line are $-\frac{2}{3}$, and $\frac{1}{3}$, what is the third? Ans. $\pm\frac{2}{3}$.

8. If a line makes angles of 60° and 45° respectively with the positive directions of the X- and Z-axes, what angle does it make with the Y-axis? Ans. 60° or 120°.

9. Find the spherical coordinates of each of the following points $(1, 2, 2)$, $(1, 4, -8)$, $(-4, -3, 0)$, $(4, -4, 2)$.

Ans. First point $r = 3$, $\theta = \tan^{-1} 2$, $\gamma = \cos^{-1} \frac{2}{3}$.
Second point $r = 9$, $\theta = \tan^{-1} 4$, $\gamma = \cos^{-1} (-\frac{8}{9})$.

10. Determine the rectangular coordinates of the point whose spherical coordinates are $r = 4$, $\theta = 60°$, $\gamma = 30°$.

Ans. $(1, \sqrt{3}, 2\sqrt{3})$.

118. Equations of the first degree containing one or two variables.

Theorem I. *Equations of the form $x = a$, $y = b$, $z = c$, where a, b, c are constants, represent* **planes** *parallel to YOZ, ZOX, XOY respectively.*

The equation $x = a$ defines a system of points all at the same distance a from YOZ. These points all lie in a plane parallel to YOZ and at the distance a from it. Hence $x = a$ is the equation of this plane.

Thus if $OM = 4$ units, the equation of the plane AMB is $x = 4$.

Similarly the theorem is proved for $y = b$ and $z = c$.

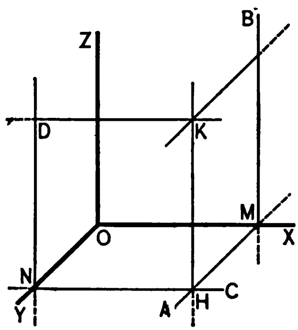


Fig. 118

Theorem II. *Two simultaneous equations of the form $x = a$, $y = b$, where a and b are constants, represent a* **straight line** *parallel to OZ.*

Let $OM = a$, $ON = b$. By Theorem I, $x = a$ represents the plane AMB, and $y = b$, the plane CND, hence the two equations taken simultaneously represent that system of points

which is common to both planes. This is the straight line HK, parallel to OZ, in which the two planes intersect.

Similarly the two pairs of equations $y = b$, $z = c$; $x = a$, $z = c$ represent straight lines parallel respectively to OX and OY.

The following important corollaries are easily deduced from Theorems I and II.

$$\text{The equation of } \boldsymbol{YOZ} \text{ is } \boldsymbol{x = 0}. \tag{9}$$

$$\text{The equation of } \boldsymbol{ZOX} \text{ is } \boldsymbol{y = 0}. \tag{10}$$

$$\text{The equation of } \boldsymbol{XOY} \text{ is } \boldsymbol{z = 0}. \tag{11}$$

$$\text{The equations of } \boldsymbol{OX} \text{ are } \boldsymbol{y = 0,\ z = 0}. \tag{12}$$

$$\text{The equations of } \boldsymbol{OY} \text{ are } \boldsymbol{z = 0,\ x = 0}. \tag{13}$$

$$\text{The equations of } \boldsymbol{OZ} \text{ are } \boldsymbol{x = 0,\ y = 0}. \tag{14}$$

THEOREM III. *The equation* $Ax + By + D = 0$ *represents a plane parallel to* OZ.

Let AB be the straight line in the plane XOY whose equation in that plane is

$$Ax + By + D = 0. \tag{i}$$

Through AB pass a plane parallel to OZ, meeting XOZ in BD, and YOZ in AC. Let P be any point in this plane $CABD$, and through P draw QP parallel to OZ, meeting AB at Q. Since QP is parallel to OZ it is also parallel to YOZ and to XOZ, so that every point on QP has the same x, and also the same y. Hence P has the same x and y as Q, and therefore the coordinates of P will satisfy equation (i) since the coordinates of Q satisfy it.

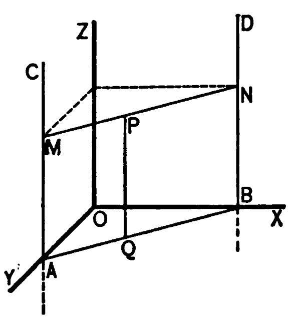


FIG. 119

Equation (i) is therefore the equation of the plane $CABD$ since it is satisfied by the coordinates of P, any point in it.

Similarly it may be shown that

$$Ax + Cz + D = 0, \qquad (ii)$$

$$By + Cz + D = 0, \qquad (iii)$$

are the equations of planes parallel respectively to OY and OX.

COROLLARY. *Any two of the three equations* (*i*), (*ii*), (*iii*), *taken simultaneously, represent a straight line.*

The two equations taken simultaneously represent that system of points common to the loci of the equations considered separately, and since both loci are in this case planes the points common to both lie in the straight line in which they intersect.

119. Equations not of the first degree containing two variables.

DEFINITIONS. *A cylindrical surface is one which is generated by a straight line which moves so that it is always parallel to a fixed straight line and always intersects a given curve which does not lie in the same plane with the fixed line, nor in a plane parallel to it.*

Every position of the generating line of a cylindrical surface is called an element of the surface.

THEOREM. *Every equation, whose degree exceeds the first, which contains only two of the three variables, x, y, z, is the equation of a cylindrical surface, whose elements are parallel to that one of the coordinate axes which corresponds to the variable missing from the equation.*

The proof is essentially the same as that of III, Art. 118. Let

$$f(x, y) = 0^* \qquad (i)$$

be any equation of higher degree than the first in x and y. In the plane XOY this equation represents some curve AB.

* See Art. 17.

Through every point of AB draw a straight line parallel to OZ. These lines will form a cylindrical surface. Take any point P in this surface. This point lies in an element of the surface PQ, and Q is the projection of P on XOY. Then as P and Q have the same x and the same y, and since the coordinates of Q satisfy (i) the coordinates of P will also satisfy it. Hence (i) is the equation of the cylindrical surface with elements parallel to OZ.

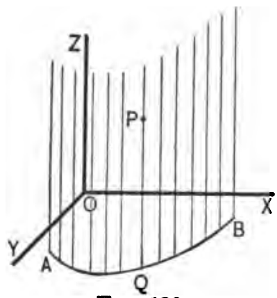


FIG. 120

Similar conclusions follow with reference to the equations $f(y, z) = 0$, $f(z, x) = 0$. Hence all equations of the form

$$f(y, z) = 0, \qquad f(z, x) = 0, \qquad f(x, y) = 0, \tag{15}$$

represent cylindrical surfaces whose elements are parallel respectively to OX, OY, OZ.

EXAMPLES. The equation $(y - 3)^2 + (z - 4)^2 = 4$, which as the equation of a plane locus represents a circle in the plane YOZ, with center at $y = 3$, $z = 4$, and radius $= 2$, represents as an equation in solid geometry a cylindrical surface with elements parallel to OX, all of whose cross-sections parallel to YOZ are equal circles.

Again the equation $x^2 + y^2 = r^2$ is that of a cylindrical surface which intersects XOY in the circle $x^2 + y^2 = r^2$, and whose axis is OZ.

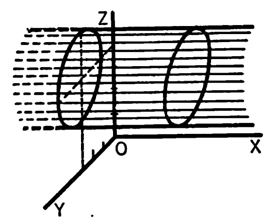


FIG. 121

EXERCISES. 1. What surfaces do the following equations represent?

(a) $x = 4$, (b) $y + 3 = 0$, (c) $z = 2$,
(d) $3y + 4 = 0$, (e) $4z - 5 = 0$, (f) $2x + 1 = 0$,
(g) $x + y = 0$, (h) $x = y$, (i) $y = mx + b$,

Ans. (*a*) A plane parallel to YOZ, 4 units on the positive side. (*g*) The plane through OZ which cuts XOY in the line $x + y = 0$. (*i*) The plane parallel to OZ, which cuts XOY in the line $y = mx + b$.

2. Write the equations of the following planes.

(*a*) Parallel to XOY, and 4 units on the positive side of it.

Ans. $z = 4$.

(*b*) Parallel to YOZ, and two units on the negative side.

(*c*) Perpendicular to OX, and $+ 4$ units from the origin.

(*d*) Parallel to ZOX, and passing through $(4, 2, 7)$.

Ans. $y - 2 = 0$.

3. What is the equation of the plane which is parallel to the Z-axis, and cuts the X- and Y-axes at $+ 2$ and $- 3$ respectively?

Ans. $3x - 2y - 6 = 0$.

4. What is the equation of the plane, parallel to the Y-axis, which passes through the points $(2, 0, 0)$ and $(0, 0, - 2)$?

5. What lines are represented by the following pairs of equations.

(*a*) $x = 3, \quad y = 2$.
(*b*) $x = - 2, \quad z = 0$.
(*c*) $y + 2 = 0, \quad z = 4$.
(*d*) $3x + 2 = 0, \quad 4y - 7 = 0$.
(*e*) $2x - 5 = 0, \quad y = 3$.
(*f*) $4y - 3 = 0, \quad z = - 2$.

Ans. (*a*) Parallel to OZ, and intersecting XOY at $(3, 2)$.
(*b*) Parallel to OY, and intersecting ZOX at $(- 2, 0)$.

6. What locus is represented by the two simultaneous equations $3x + 4y - 7 = 0$, $4x - 2y - 1 = 0$. Construct the locus.

7. Describe the surfaces represented by the following equations, and make drawings of them.

(*a*) $y^2 = 4x$,
(*b*) $x^2 + y^2 = 25$,
(*c*) $4z^2 + 9y^2 = 36$,
(*d*) $4y^2 = x^3$,
(*e*) $x^2 - 4 = 0$,
(*f*) $4z^2 - 9x^2 = 36$,
(*g*) $y = \sin x$,
(*h*) $x^2 + z^2 + 4x - 6z - 12 = 0$.

Ans. (*a*) A parabolic cylinder with elements parallel to OZ.
(*b*) A circular cylinder with elements parallel to OZ.

120. Equation of the first degree in three variables.

THEOREM. *Every equation of the first degree in three variables*

$$Ax + By + Cz + D = 0 \tag{16}$$

is the equation of a plane.

Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ be any two points on the locus represented by equation (16). Then

$$Ax_1 + By_1 + Cz_1 + D = 0, \qquad (i)$$
$$Ax_2 + By_2 + Cz_2 + D = 0. \qquad (ii)$$

Also let $P_3(x_3, y_3, z_3)$ be any other point on the straight line through the points P_1, P_2 If it can be shown that P_3 also lies in the locus whose equation is (16), then this locus must be a plane, because by elementary geometry a plane is a surface such that if a straight line be drawn through any two of its points then every point of this line lies in the surface.

PROOF. Since by hypothesis P_1, P_2, P_3 are in the same straight line the direction cosines of P_1P_3 and those of P_1P_2 must be equal, each to each. Let the length of $P_1P_2 = d$, and that of $P_1P_3 = d'$. Then by (4), p. 210,

$$\frac{x_3 - x_1}{d'} = \frac{x_2 - x_1}{d}, \qquad \text{or} \qquad d(x_3 - x_1) = d'(x_2 - x_1), \qquad (iii)$$

and, similarly

$$d(y_3 - y_1) = d'(y_2 - y_1), \qquad (iv)$$
$$d(z_3 - z_1) = d'(z_2 - z_1). \qquad (v)$$

Hence, multiplying (*iii*) by A, (*iv*) by B, and (*v*) by C

$$\left.\begin{aligned} Ad(x_3 - x_1) &= Ad'(x_2 - x_1), \\ Bd(y_3 - y_1) &= Bd'(y_2 - y_1), \\ Cd(z_3 - z_1) &= Cd'(z_2 - z_1). \end{aligned}\right\} \qquad (vi)$$

Adding the three equations (*vi*),

$$\begin{aligned} d(Ax_3 + By_3 + Cz_3) &- d(Ax_1 + By_1 + Cz_1) \\ &= d'(Ax_2 + By_2 + Cz_2) - d'(Ax_1 + By_1 + Cz_1). \qquad (vii) \end{aligned}$$

But from (*i*) and (*ii*)

$$Ax_1 + By_1 + Cz_1 = -D, \qquad Ax_2 + By_2 + Cz_2 = -D,$$

hence substituting these values, (*vii*) reduces to

$$d(Ax_3 + By_3 + Cz_3) + dD = d'(D - D) = 0$$

or

$$Ax_3 + By_3 + Cz_3 + D = 0.$$

That is, the coordinates (x_3, y_3, z_3) satisfy equation (16), hence P_3 lies in the locus of which this is the equation, and that locus is therefore a plane.

121. The equation of the sphere.

DEFINITION. *A* ***sphere*** *is the locus of all points in space at a constant distance from a fixed point. The fixed point is called the* ***center*** *of the sphere, and the constant distance, the* ***radius.***

The equation is obtained by translating the definition into algebraic language. Let (x_0, y_0, z_0) be the center, (x, y, z) any point on the locus, and r the radius. Then by (1), p. 207, the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \qquad (17)$$

expresses that the distance between the points (x_0, y_0, z_0) and (x, y, z) is always constant. Hence (17) is the equation of a sphere of radius r, with center at (x_0, y_0, z_0).

If the origin be taken at the center of the sphere, $x_0 = y_0 = z_0 = 0$, and equation (17) reduces to

$$x^2 + y^2 + z^2 = r^2. \qquad (18)$$

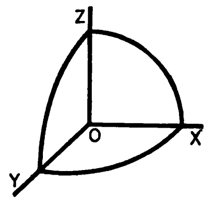


FIG. 122

Fig. 122 represents one eighth of the sphere whose equation is (18).

122. Surfaces and curves.—In Arts. 120, 121 it has been shown that two well-known surfaces, the plane and sphere, are represented in analytic geometry by equations in three variables. Similarly other surfaces may be represented by other forms of equations in three variables. The relation between an equation in three variables and a surface is the same as that between an equation in two variables and a locus in a plane.

The notation $f(x, y, z)$, see Art. 17, is used to designate any expression containing the three variables x, y, z. Thus in general

$$f(x, y, z) = 0 \tag{19}$$

is the equation of a surface.

NOTE. The equation $f(x, y, z) = 0$ includes equations of the type $x^2 + y^2 - a^2 = 0$, $y - b = 0$, etc., in which only one or two variables occur explicitly. These latter as the equations of loci in space are interpreted as equations in three variables in which the terms containing the missing variables have the coefficient zero.

THEOREM. *Two simultaneous equations in three variables, x, y, z, represent one or more lines, straight or curved.*

Let

$$f(x, y, z) = 0, \qquad \phi(x, y, z) = 0,$$

be the equations of two surfaces. The points whose coordinates satisfy both equations are those points which lie in both surfaces. The two equations taken simultaneously therefore represent the lines straight or curved in which the two surfaces intersect.

123. Interpretation of equations.—A method will now be described for determining the appearance and situation of a surface whose equation is given. A preliminary discussion will help to make clear the nature of the process.

I. Let

$$f_1(x, y, z) = 0, \qquad f_2(x, y, z) = 0 \tag{i}$$

be the equations of two surfaces. Let AB represent the curve of intersection of these surfaces. If one of the variables, for example z, be eliminated from the equations (i) the result will be an equation

$$\phi(x, y) = 0, \tag{ii}$$

which is satisfied by the coordinates of every point on the curve AB in which the surfaces (i) intersect. By Art. 119, equation (ii)

is the equation of a cylinder whose elements are parallel to the Z-axis, and hence it must be the equation of the cylinder, with elements P_1Q_1, P_2Q_2, P_3Q_3, etc., parallel to OZ, upon which

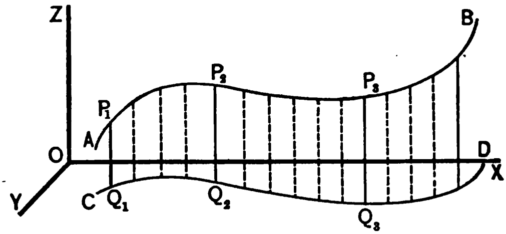


FIG. 123

lies the curve of intersection of the surfaces (i). It is therefore also the equation in the plane XOY of the curve CD, in which this cylinder intersects XOY, that is the equation of the projection of AB on XOY.

Similar results follow if y or x be eliminated from the two equations (i).

II. If one of the two surfaces given by equations (i) is a plane parallel to XOY, the curve of intersection AB, and its projection CD on XOY will be identical (see Art. 114, IV). The equation (ii) in this case may be taken either as the equation of CD in XOY, or as the equation of the curve of intersection itself in the cutting plane. Similar results follow for intersections of a surface by planes parallel to YOZ or XOZ.

III. The form of a surface whose equation is given can often be outlined by determining its intersections with the coordinate planes, and with planes parallel to them, as indicated in II. For example, every plane parallel to XOY has for its equation $z = k$, a constant. If k be substituted for z in the equation of the surface, the resulting equation in x and y is the equation in the plane $z = k$ of the curve in which this plane cuts the surface. This method will be illustrated by examples.

1. Construct the surface $4x + 3y + 5z = 12$.

By Art. 120 this is the equation of a plane. The intersection with the plane XOY is obtained by combining the given equation with $z = 0$, the equation of XOY. This gives

$$4x + 3y = 12,$$

which is therefore the equation in the plane XOY of the intersection, the line AB.

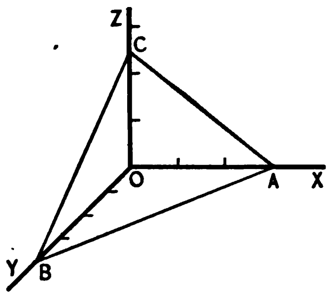


Fig. 124

Similarly, making $y = 0$, the result $4x + 5z = 12$ is the equation of CA, the intersection of the given plane with ZOX; and making $x = 0$, the intersection BC of the given plane with YOZ is found to be $3y + 5z = 12$. These three intersections show the position of the plane.

2. Construct the surface

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$$

Cutting the surface by the plane XOY, by making $z = 0$ in the given equation, the result is the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1,$$

whose semi-axes are $OA = 5$, $OB = 4$.

Similarly making $y = 0$, the section formed by the plane ZOX is the ellipse

$$\frac{x^2}{25} + \frac{z^2}{9} = 1,$$

whose semi-axes are $OA = 5$, $OC = 3$.

Finally, making $x = 0$, the section formed by the plane YOZ is the ellipse

$$\frac{y^2}{16} + \frac{z^2}{9} = 1,$$

whose semi-axes are $OB = 4$, $OC = 3$.

Only one eighth of the figure is drawn, as it is evident that the surface is symmetrical with respect to the origin.

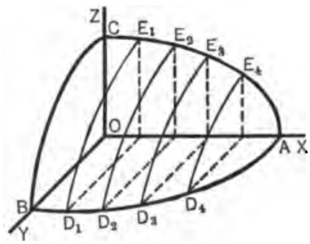


FIG. 125

Sections of the surface formed by planes parallel to YOZ are obtained by combining the given equation with the equations of such planes, which all have the form $x = k$, where k is a constant. Substituting $x = k$ the given equation becomes

$$\frac{y^2}{16} + \frac{z^2}{9} = 1 - \frac{k^2}{25},$$

or

$$\frac{y^2}{16\left(1 - \frac{k^2}{25}\right)} + \frac{z^2}{9\left(1 - \frac{k^2}{25}\right)} = 1.$$

This is the equation of a real ellipse in the plane $x = k$ for all values of $k^2 < 25$, but if $k^2 > 25$ the curve is imaginary. Hence all sections of the surface parallel to YOZ are ellipses, and the surface does not extend beyond A, where $k = 5$. D_1E_1, D_2E_2, etc., represent quadrants of some of these ellipses.

Similarly by assigning constant values to y and z in turn it is shown that sections of the surface parallel to ZOX and XOY respectively, are ellipses, and that the surface does not extend beyond B, where $OB = 4$ in the direction OY, nor beyond C, where $OC = 3$ in the direction OZ.

Similar results may be obtained for any surface whose equation is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{20}$$

or for any surface whose equation can be reduced to that form, such as

$$Ax^2 + By^2 + Cz^2 = D, \tag{21}$$

where A, B, C, D are positive constants.

This surface is called an **ellipsoid,** and the three distances $OA = a$, $OB = b$, $OC = c$, Fig. 125, are called the **semi-axes** of the ellipsoid.

If the three semi-axes, a, b, c, of an ellipsoid are equal the surface is a sphere. See equation (18), p. 219.

If two of the semi-axes, as b and c are equal, equation (20) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1, \tag{22}$$

and the series of sections of this surface determined by the planes $x = k$ have equations of the form

$$y^2 + z^2 = b^2\left(1 - \frac{k^2}{a^2}\right),$$

which are circles. The surface represented by equation (22) is therefore a **surface of revolution.** It may be conceived as being generated by the rotation of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ about its major-axis.

An ellipsoid of revolution is usually called a **spheroid.** If the two equal axes are shorter than the third axis, that is if $b < a$ in (22) the surface is called a **prolate spheroid,** but if the two equal axes are longer than the third, or $b > a$ in (22), the surface is an **oblate spheroid.**

3. Construct the surface

$$\frac{x^2}{25} + \frac{y^2}{16} - \frac{z^2}{9} = 1.$$

The analysis of this equation is made as in Ex. 2. The section by the plane XOY, or $z = 0$, is the ellipse AB

$$\frac{x^2}{25} + \frac{y^2}{16} = 1,$$

whose semi-axes are $OA = 5$, $OB = 4$.

The section by the plane ZOX, or $y = 0$, is the hyperbola AC

$$\frac{x^2}{25} - \frac{z^2}{9} = 1,$$

which cuts OX at A, where $OA = 5$.

The section by the plane YOZ, or $x = 0$, is the hyperbola BD

$$\frac{y^2}{16} - \frac{z^2}{9} = 1,$$

which cuts OY at B, where $OB = 4$.

Only one fourth of each of these sections is shown in the figure.

Sections parallel to XOY, obtained by giving to z a series of constant values, positive or negative, are always real ellipses. Thus if $z = \pm k$ the equations of these sections in their respective planes are

$$\frac{x^2}{25} + \frac{y^2}{16} = 1 + \frac{k^2}{9}.$$

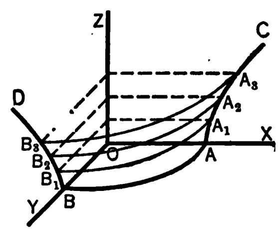


FIG. 126

Quadrants of these elliptic sections are shown at A_1B_1, A_2B_2, etc.

Sections parallel to YOZ, and to ZOX, obtained by making $x = k$ and $y = k$, respectively are in general hyperbolas, as the

student can show, but there are important exceptions to this general statement.

Thus if the surface be cut by a plane through B, by making $y = 4$, the result is $\frac{1}{25}x^2 - \frac{1}{9}z^2 = 0$, or

$$\tfrac{1}{5}x \pm \tfrac{1}{3}z = 0,$$

which represents in the plane $y = 4$ a pair of straight lines meeting at B. Similarly, making $x = 5$, it is seen that this plane cuts the surface in a pair of straight lines meeting at A, whose equations in the plane $x = 5$ are $\frac{1}{4}y \pm \frac{1}{3}z = 0$. Similar results are given by the two planes $y = -4$, and $x = -5$.

Similar results may be obtained for any surface whose equation has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad (23)$$

or, more generally, whose equation has the form

$$Ax^2 + By^2 + Cz^2 = D, \qquad (24)$$

where one of the three letters A, B, C stands for a negative number and the other two, with D, for positive numbers.

This surface is called a **hyperboloid of one sheet.**

If $a = b$ the sections A_1B_1, A_2B_2, etc., are circles, and equation (23) then represents a **hyperboloid of revolution of one sheet.** It can be produced by rotating the hyperbola $(x^2/a^2) - (z^2/c^2) = 1$ about its conjugate axis.

4. Construct the surface

$$\frac{x^2}{9} - \frac{y^2}{16} - \frac{z^2}{25} = 1.$$

If this surface be cut by the plane XOY by substituting $z = 0$ in the given equation, the section is the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{16} = 1,$$

AB, $A'B'$ in the figure, in which $OA = 3$. Similarly the section by the plane ZOX is the hyperbola AC, $A'C'$, with the same transverse axis.

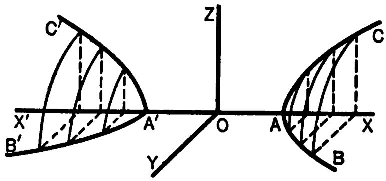


Fig. 127

If $x = 0$ the equation is $-(y^2/16) - (z^2/25) = 1$, showing that the section by the plane YOZ is imaginary. If, however, sections parallel to YOZ be formed by making $x = k$, a constant, the resulting equation is

$$\frac{y^2}{16\left(\frac{k^2}{9} - 1\right)} + \frac{z^2}{25\left(\frac{k^2}{9} - 1\right)} = 1,$$

which represents a real ellipse in the plane $x = k$ if $k > 3$, or < -3. Hence all sections parallel to YOZ are real ellipses if the cutting plane is more than three units from YOZ on either side.

Similar results may be obtained for any surface whose equation has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \tag{25}$$

or,

$$Ax^2 + By^2 + Cz^2 = D \tag{26}$$

where two of the three letters A, B, C stand for negative numbers, and the other, with D, for positive numbers.

This surface is called the **hyperboloid of two sheets.**

If $b = c$ in equation (25) the sections parallel to YOZ are circles, and the surface is a **hyperboloid of revolution of two sheets.** It may be produced by rotating the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ about its transverse axis.

5. Construct the surface

$$\frac{x^2}{25}+\frac{y^2}{9}=\frac{z}{4}.$$

Cutting the surface by XOY by making $z = 0$, the result is

$$\tfrac{1}{25}x^2+\tfrac{1}{9}y^2=0.$$

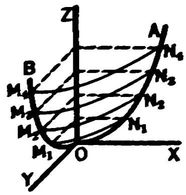


FIG. 128

This equation is satisfied by only one pair of real values of x and y, $x = 0$, $y = 0$. Hence the surface has only one real point, the origin, in the plane XOY.

Making $y = 0$, the section by the plane ZOX is found to be

$$x^2=\tfrac{25}{4}z,$$

a parabola OA tangent to OX at the vertex O.

Similarly, making $x = 0$, the section by the plane YOZ is

$$y^2=\tfrac{9}{4}z,$$

a parabola OB tangent to OY at the vertex O.

To cut the surface by planes parallel to XOY give to z a series of constant values $z = k$. Thus we have

$$\frac{x^2}{25}+\frac{y^2}{9}=\frac{k}{4},$$

which is always an ellipse in the plane $z = k$, real if k is positive, and imaginary if k is negative. M_1N_1, M_2N_2, etc., represent quadrants of these elliptic sections.

The student may show that sections of this surface parallel to YOZ, or to ZOX, are always parabolas.

The general equation of this type is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c} \tag{27}$$

or

$$Ax^2 + By^2 = Cz, \tag{28}$$

where A and B are positive. This surface is called the **elliptic paraboloid.** If $a = b$ in (27) it becomes a **paraboloid of revolution,** that is the surface produced by revolving a parabola about its axis.

6. Construct the surface

$$\frac{x^2}{25} - \frac{y^2}{16} = \frac{z}{3}.$$

Making $z = 0$, the section formed by the plane XOY is seen to be a pair of straight lines $(x^2/25) - (y^2/16) = 0$, or $(x/5) \pm (y/4) = 0$, one of which, OB, is drawn in the figure.

Making $y = 0$, the section formed by the plane ZOX is found to be the parabola OC,

$$x^2 = \tfrac{25}{3} z.$$

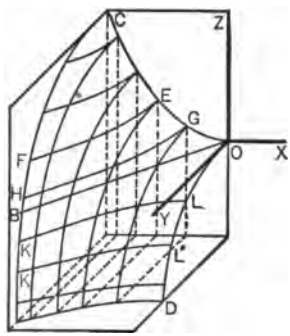


FIG. 129

Making $x = 0$ the section by the plane YOZ is the parabola OD,

$$y^2 = -\tfrac{16}{3} z.$$

The series of planes $z = k$, parallel to XOY, form sections which are always hyperbolas

$$\frac{x^2}{25} - \frac{y^2}{16} = \frac{k}{3}.$$

The transverse axes of these hyperbolas are parallel to OX if k is positive, and parallel to OY if k is negative. The hyperbolas GH and EF, represent portions of two of these sections for

positive values of k, and KL, $K'L'$ represent portions of these hyperbolic sections for negative values of k.

For all planes parallel to XOZ, given by $y = k$, the sections are parabolas with axes extending upward, and for all planes parallel to YOZ, given by $x = k$, the sections are parabolas with axes extending downward.

The general equation of this type is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c} \tag{29}$$

or

$$Ax^2 + By^2 = Cz, \tag{30}$$

where A and B represent numbers having opposite signs.

This surface is called a **hyperbolic paraboloid.** It has quite important uses and applications. Its general appearance is best studied with the aid of a solid model.

7. Construct the surface

$$\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{25} = 0.$$

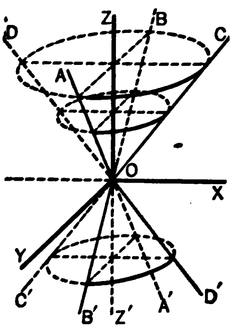


FIG. 130

By methods fully illustrated in the preceding examples it can be shown that the plane XOY cuts the surface in a point (the origin), and that ZOX, and YOZ cut it respectively in the pairs of straight lines $(x/4) \pm (z/5) = 0$, or OC, OD, and $(y/3) \pm (z/5) = 0$, or OA, OB.

The sections parallel to XOY, found by making $z = k$, are ellipses for all values of k, positive and negative.

Let the surface be intersected by any plane $y = mx$ passing through OZ. Eliminating y between this and the given equation the result is

$$x^2(\tfrac{1}{16} + \tfrac{1}{9}m^2) - \tfrac{1}{25}z^2 = 0.$$

By I, p. 220, this equation is that of the projection of the intersection of $y = mx$ with the surface on the plane XOZ. From the form of the equation it is seen that this projection is a pair of straight lines meeting at O. If x be eliminated between $y = mx$ and the given equation the result is

$$y^2\left(\frac{1}{16m^2}+\frac{1}{9}\right)-\frac{1}{25}z^2 = 0,$$

showing that the projection of the intersection on YOZ is also a pair of straight lines meeting at the origin. Since both of these projections on perpendicular planes consist of pairs of straight lines the intersection itself must be a pair of straight lines, for all values of m. That is, every plane through OZ cuts the surface in a pair of straight lines meeting at O. The surface is therefore a **cone**, with vertex at O, and axis coincident with OZ.

NOTE. A *conical surface*, or a *cone*, is a surface generated by a straight line moving in space so that it always passes through a fixed point called the *vertex*, and always intersects a given curve which does not lie in the same plane with the vertex. Every position of the moving line is called an element of the cone.

The general equation of this type is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 0, \tag{31}$$

or

$$Ax^2 + By^2 + Cz^2 = 0, \tag{32}$$

where A, B, and C do not all have the same sign. The cone represented by (32) will have its axis coincident with OX, OY, or OZ, according as the sign of A, B or C differs from that of the other two of these letters.

If in equation (31) $a = b$ the surface is a **right circular cone**, or a **cone of revolution.**

124. Concluding remarks.—This chapter is intended to give only the briefest possible introduction to some of the principles of coordinate geometry of space, especially those which will be needed when, in the study of calculus, the student comes to applications in which solids and the surfaces of solids are treated. A fuller introduction to this interesting, important, and extensive branch of mathematics may be obtained from the special elementary treatises on the subject.

EXERCISES ON CHAPTER XII

Normal Exercises

1. Locate on a figure the points whose rectangular coordinates are $(1, 3, 2)$ and $(-4, 2, -3)$.

2. Locate the points whose spherical coordinates are $r = 4$, $\theta = 60°$, $\gamma = 30°$, and $r = 2$, $\theta = 225°$, $\gamma = 150°$.

3. Find the lengths of the sides of the triangle whose vertices are the points $(4, 2, -3)$, $(2, 1, -1)$, $(-2, -3, 1)$.

Ans. $3, 6, \sqrt{77}$.

4. Find the lengths of the projections on the three coordinate axes of the line joining $(-2, 4, 5)$ and $(3, -8, 5)$.

5. Find the direction cosines of the line joining (*a*) the origin to $(1, -4, 4)$, (*b*) $(-2, 4, 5)$ to $(3, -8, 5)$.

Ans. (*a*) $1 : \sqrt{33}, -4 : \sqrt{33}, 4 : \sqrt{33}$. (*b*) $\frac{5}{13}, -\frac{12}{13}, 0$.

6. A line joining the origin and a point whose coordinates are all positive makes with the X-axis an angle of 40°, and with the Z-axis an angle of 65°. What angle does it make with the Y-axis?

Ans. $61° 2'$.

7. Determine the spherical coordinates of the point whose rectangular coordinates are $(4, -4, -2)$.

Ans. $r = 6$, $\theta = 315°$, $\gamma = \cos^{-1}(-\frac{1}{3})$.

8. If the spherical coordinates of a point are $r = 3$, $\theta = 120°$, $\gamma = 135°$, what are its rectangular coordinates?

Ans. $(-\frac{3}{4}\sqrt{2}, \frac{3}{4}\sqrt{6}, -\frac{3}{2}\sqrt{2})$.

9. Determine the surfaces represented by the following equations

(*a*) $x = 4$, (*b*) $x^2 - y^2 = 0$, (*c*) $z - 3x + 5 = 0$,
(*d*) $y = -2$, (*e*) $4x + y = 0$,
(*f*) $4x + 3y + z - 24 = 0$, (*g*) $x^2 + y^2 = 4$,
(*h*) $3x - 4y + 2z - 5 = 0$, (*i*) $x^2 + y^2 + z^2 - 25 = 0$,
(*j*) $4x^2 + 4y^2 + 4z^2 - 8x + 12z - 3 = 0$.

10. Find the equations of the following spheres

(*a*) center at the origin, radius $= 5$,
(*b*) center at $(-1, 2, -3)$, radius $= 4$.

11. Construct the loci represented by the following pairs of equations

(*a*) $x = 2, \quad y = 4$. (*d*) $x + 3y - 7 = 0, \quad y = 4$.
(*b*) $x + 3 = 0, \quad z = 4$. (*e*) $x + 3y - 7 = 0, \quad z = 4$.
(*c*) $x^2 + y^2 = 16, \quad z = 4$. (*f*) $x^2 - 4y^2 = 9, \quad z = 0$.

12. Discuss and sketch the following surfaces

(*a*) $4x^2 + 9y^2 + z^2 - 36 = 0$, (*d*) $4z^2 + 9y^2 - x = 0$,
(*b*) $4x^2 + 4y^2 - z^2 - 16 = 0$, (*e*) $y^2 - 4x^2 - 4z = 0$,
(*c*) $z^2 - 4x^2 - y^2 - 64 = 0$, (*f*) $4x^2 - 9y^2 + 9z^2 = 0$.

General Exercises

13. In the case of the following planes find their intercepts on the coordinate axes, and the equations of the lines in which they intersect the coordinate planes.

(*a*) $4x + 3y + 7z - 12 = 0$, (*d*) $x + y + z + 1 = 0$,
(*b*) $2x - y + 8z - 16 = 0$, (*e*) $x + y + z = 0$,
(*c*) $8x + 3y - 4z = 0$, (*f*) $3x + 4y - 2z - 12 = 0$.

Ans. (*a*) Intercepts are 3, 4, $\frac{12}{7}$; intersections with YOZ, ZOX, XOY, respectively are $3y + 7z - 12 = 0$, $4x + 7z - 12 = 0$, $4x + 3y - 12 = 0$.

14. Find the equation of the plane which passes through each of the following sets of points

(*a*) $(1, 3, 4), (2, 4, 3), (4, 1, 1)$. (*c*) $(5, 0, 0), (0, 5, 0), (0, 0, 5)$.
(*b*) $(-2, 3, 5), (1, 0, -3), (3, 2, 0)$. (*d*) $(0, 0, 0), (0, 0, 1), (0, 5, 0)$.

Ans. (*a*) $x + z - 5 = 0$, (*b*) $7x - 25y + 12z + 29 = 0$.

15. Find the equations of the planes determined by the following sets of conditions.

(*a*) Passing through the X-axis, and the point (4, 5, 6).
Ans. $6y - 5z = 0$.

(*b*) Passing through the points $(2, -1, 3)$, $(3, 1, -2)$, and parallel to the Z-axis. Ans. $2x - y - 5 = 0$.

(*c*) Passing through the line $x = 2$, $y = 3$, and the point (3, 1, 7).
Ans. $2x + y - 7 = 0$.

(*d*) Passing through the line $3x + 4y - 7 = 0$, $z = 0$, and the point $(5, -2, 3)$. Ans. $3x + 4y - 7 = 0$.

(*e*) Passing through the line $3x + 4y - 5z - 12 = 0$, $4x + 2y + 5z + 7 = 0$, and parallel to the Z-axis. Ans. $7x + 6y - 5 = 0$.

16. Find the equation of the sphere

(*a*) whose center is at (2, 1, 3), and radius 4,

(*b*) whose center is at (0, 0, 0), and radius 5,

(*c*) with center at (1, 2, 3), and passing through (3, 1, 1),

(*d*) with center at (4, 3, 5), and tangent to XOY.

Ans. (*a*) $x^2 + y^2 + z^2 - 4x - 2y - 6z - 2 = 0$,
(*c*) $x^2 + y^2 + z^2 - 2x - 4y - 6z + 5 = 0$.

17. Determine the centers and radii of the following spheres.

(*a*) $x^2 + y^2 + z^2 - 4x + 2y - 8z - 4 = 0$,
(*b*) $4x^2 + 4y^2 + 4z^2 + 8x - 4y + 12z + 65 = 0$.
Ans. (*a*) Center, $(2, -1, 4)$, $r = 5$. (*b*) Imaginary sphere.

18. Find the equation of the spheres determined by the following sets of conditions.

(*a*) Tangent to the XY plane, center at (4, 5, 6).
Ans. $x^2 + y^2 + z^2 - 8x - 10y - 12z + 41 = 0$.

(*b*) Tangent to the X-axis, center at (2, 3, 4).
Ans. $x^2 + y^2 + z^2 - 4x - 6y - 8z + 4 = 0$.

(*c*) Center at (2, 3, 4), and tangent to the plane $x = 5$.

(*d*) Passing through the four points $(2, 2, -4)$, $(-2, 6, 0)$, $(6, -2, 4)$, $(4, 6, 6)$. Ans. $x^2 + y^2 + z^2 - 4x - 4y - 4z - 24 = 0$.

19. Discuss and sketch the following surfaces

(a) $4x^2 + 9y^2 + z^2 - 36 = 0$,
(b) $x^2 + y^2 - z^2 = 36$,
(c) $4x^2 + y^2 - 9z^2 = 36$,
(d) $4x^2 + 4y^2 - 9z = 0$,
(e) $3x + 4y + z = 12$,
(f) $y^2 - 4x^2 - 25z^2 = 4$,
(g) $9x^2 - 36y^2 + 25z^2 = 225$,
(h) $4x^2 + 4y^2 + 4z^2 - 25 = 0$,
(i) $x^2 + y^2 - 4z = 4$,
(j) $z^2 - 9y^2 - 16x^2 = -36$,
(k) $4x^2 + 9y^2 + z^2 + 4 = 0$,
(l) $z^2 - 4y^2 - 4x = 0$,
(m) $4y^2 + 9x^2 + 36z + 36 = 0$,
(n) $x^2 - y^2 + z^2 = 0$,
(o) $4x^2 + 4y^2 - 25z^2 = 0$,
(p) $z^2 + 3y^2 - 27x^2 = 0$.

Ans. (a) Ellipsoid, $a = 3$, $b = 2$, $c = 6$. (b) Hyperboloid of revolution of one sheet, $a = b = c = 6$. (c) Hyperboloid of one sheet, $a = 3$, $b = 6$, $c = 2$. (d) Paraboloid of revolution, with axis in OZ.

20. Discuss and sketch the following surfaces. Note the changes produced by changes in value and sign of the coefficients.

(a) $4x^2 + 9y^2 + z^2 - 36 = 0$,
(b) $9x^2 + 4y^2 + z^2 - 36 = 0$,
(c) $9x^2 + 4y^2 + z^2 + 36 = 0$,
(d) $9x^2 + 4y^2 - z^2 - 36 = 0$,
(e) $9x^2 - 4y^2 - z^2 = 0$,
(f) $9x^2 + 4y^2 - z^2 = 0$,
(g) $9x^2 + 4y^2 - z = 0$,
(h) $9x^2 + 4y^2 - z - 36 = 0$,

21. If the direction cosines of a line are to each other as $l : m : n$, what are their values?

22. Find the coordinates of the middle point of the line joining $(3, -4, 6)$ and $(7, 2, -2)$. Ans. $(5, -1, 2)$.

23. Derive formulas for the coordinates of the middle point of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2).

24. Find the equation of the surface every point of which is equally distant from $(4, 3, 7)$ and $(2, 4, 3)$. Describe and sketch the surface.
Ans. $4x - 2y + 8z - 45 = 0$.

25. Find the equation of the locus of a point whose distance from $(4, 0, 0)$ is always twice its distance from $(1, 0, 0)$. Describe and sketch the surface. Ans. $x^2 + y^2 + z^2 = 4$.

26. A point moves in space so that the sum of its distances from $(2, 0, 0)$ and $(-2, 0, 0)$ is always equal to 8. Describe and sketch its locus. Ans. $3x^2 + 4y^2 + 4z^2 = 48$.

APPENDIX

CURVES FOR REFERENCE

A number of curves, not referred to in the preceding pages, which are either important practically, or are of interest historically, are gathered here for convenience of reference.

In the equations which follow a and b always stand for given arbitrary lengths, k, for a numerical constant. When an equation contains one arbitrary length a or b, a change in its value simply alters the scale of the drawing. A change in the value of a numerical constant, however, or a change in the relative values of two arbitrary lengths, may change completely the appearance of the curve.

The *cubical parabola*, $y = x^3$ or $a^2y = x^3$.

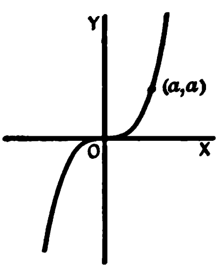


FIG. 131

The *semi-cubical parabola*, $y^2 = x^3$, or $ay^2 = x^3$.

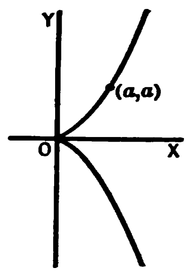


FIG. 132

The *cissoid*.

$$y^2 = \frac{x^3}{2a - x}, \qquad r = \frac{2a \sin^2 \theta}{\cos \theta}.$$

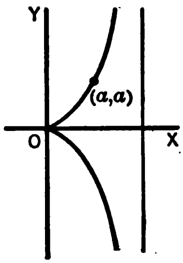


FIG. 133

The *parabola*.

$$x^{1/2} + y^{1/2} = a^{1/2}.$$

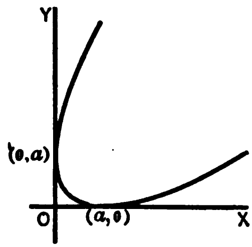


FIG. 134

The *witch*.

$$y = \frac{8a^3}{4a^2 + x^2}.$$

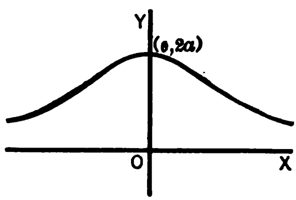


FIG. 135

The *probability curve*.

$$y = ae^{-k^2x^2}$$

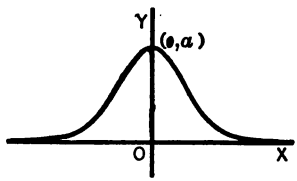
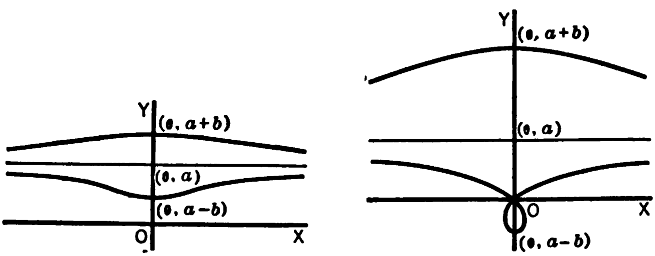


FIG. 136

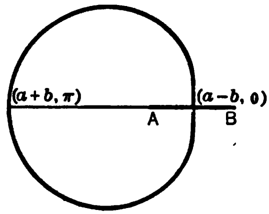
The *conchoid*. $(x^2 + y^2)(y - a)^2 = b^2y^2$. $r = a \csc \theta + b$.

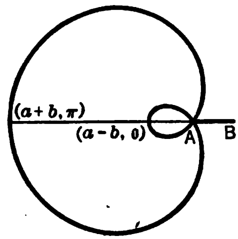


$a > b$ $a < b$

FIG. 137

The *limaçon*. $r = a - b \cos \theta$.





$a > b$ Fig. 138 $a < b$

The *cardioid*.

$$r = a(1 - \cos \theta).$$

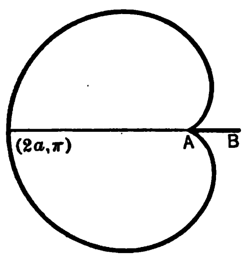


Fig. 139

The *catenary*.

$$y = \tfrac{1}{2}a(e^{x/a} + e^{-x/a}).$$

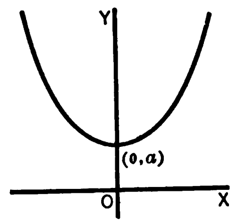


Fig. 140

The *hyperbolic spiral*, or *reciprocal spiral*.

$$r\theta = a.$$

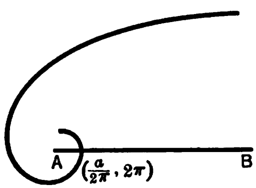


Fig. 141

The *logarithmic spiral*.

$$r = a^{\theta}.$$

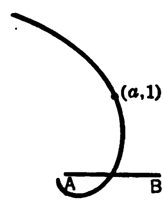


Fig. 142

The *tangent curve*.

$y = \tan x$.

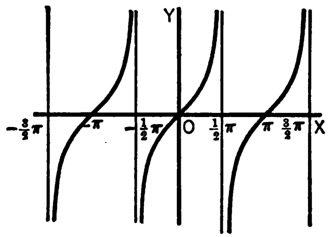


FIG. 143

The *secant curve*.

$y = \sec x$.

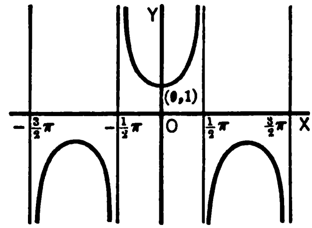


FIG. 144

$r = a \sin 2\theta$.

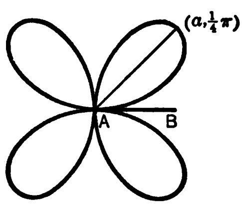


FIG. 145

$r = a \tan 3\theta$.

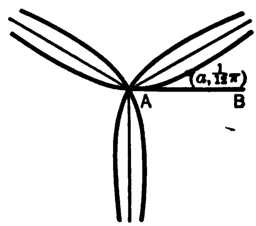
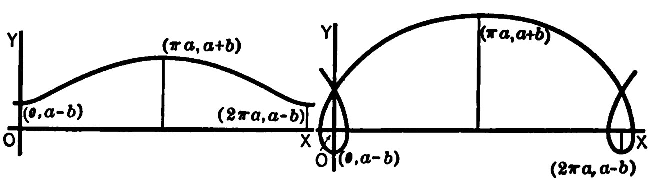


FIG. 146

The *trochoid*. $x = a\theta - b \sin \theta$. $y = a - b \cos \theta$.



$a > b$ FIG. 147 $a < b$

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